

A Smoothing Newton Algorithm for the LCP with a Sufficient Matrix that Terminates Finitely at a Maximally Complementary Solution

Jie Sun* and Zheng-Hai Huang[†]

November 21, 2004

Abstract

By using a smoothing function, the linear complementarity problem (LCP) can be reformulated as a parameterized smooth equation. A Newton method with a projection-type testing procedure is proposed to solve this equation. We show that, for the LCP with a sufficient matrix, the iteration sequence generated by the proposed algorithm is bounded as long as the LCP has a solution. This assumption is weaker than the ones used in most existing smoothing algorithms. Moreover, we show that the proposed algorithm can find a maximally complementary solution to the LCP in a finite number of iterations.

Keywords Linear complementarity problem, smoothing method, maximally complementary solution, sufficient matrix, finite termination.

AMS subject classifications 90C33, 65K10.

*Department of Decision Sciences, National University of Singapore, Singapore 119260, Republic of Singapore. Email: jsun@nus.edu.sg. Fax: (65)67792621.

[†]Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, P.R. China. Email: zhenghaihuang@yahoo.com.cn. This author's work was partially supported by the Scientific Research Foundation of Tianjin University for the Returned Overseas Chinese Scholars and the Scientific Research Foundation of Liu Hui Center for Applied Mathematics, Nankai University-Tianjin University.

1 Introduction

Given a matrix $M \in \mathfrak{R}^{n \times n}$ and a vector $q \in \mathfrak{R}^n$, the linear complementarity problem (LCP), denoted by $\text{LCP}(M, q)$, is to find a vector $(x, s) \in \mathfrak{R}^{2n}$ such that

$$(x, s) \geq 0, \quad s = Mx + q, \quad \text{and} \quad x^T s = 0.$$

Throughout this paper, the feasible set and solution set of $\text{LCP}(M, q)$ are denoted respectively by \mathcal{F} and \mathcal{S} , i.e.,

$$\mathcal{F} := \{(x, s) \in \mathfrak{R}^{2n} : s = Mx + q\} \quad \text{and} \quad \mathcal{S} := \{(x, s) \in \mathcal{F} : (x, s) \geq 0 \text{ and } x^T s = 0\},$$

where “:=” means “is defined as”. We suppose in this paper that the matrix M is a sufficient matrix. Let $\mathcal{I} := \{1, 2, \dots, n\}$ and let u_i be the i -th component of a vector u . A matrix $M \in \mathfrak{R}^{n \times n}$ is said to be *column sufficient* if for all $x \in \mathfrak{R}^n$

$$x_i(Mx)_i \leq 0, \quad i \in \mathcal{I} \quad \implies \quad x_i(Mx)_i = 0$$

and *row sufficient* if M^T is column sufficient. M is said to be *sufficient* if it is both column and row sufficient. Väliäho [29] proved that the class of sufficient matrices coincides with the class of P_* -matrices. A matrix M is called a P_* -matrix if there exists a nonnegative number κ such that

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+(x)} x_i(Mx)_i + \sum_{i \in \mathcal{I}_-(x)} x_i(Mx)_i \geq 0, \quad \forall x \in \mathfrak{R}^n, \quad (1.1)$$

where

$$\mathcal{I}_+(x) := \{i \in \mathcal{I} : x_i(Mx)_i > 0\} \quad \text{and} \quad \mathcal{I}_-(x) := \{i \in \mathcal{I} : x_i(Mx)_i < 0\}.$$

The set of all matrices satisfying (1.1) is denoted by $P_*(\kappa)$ so that $P_* = \cup_{\kappa \geq 0} P_*(\kappa)$. Obviously, $P_*(0)$ is the set of positive semidefinite matrices. Some results in this paper involves the class of P_0 matrices. A matrix M is called a P_0 -matrix if for any $x \in \mathfrak{R}^n$ with $x \neq 0$, there exists an index i such that $x_i \neq 0$ and $x_i(Mx)_i \geq 0$. It is easily seen that $P_*(0) \subset P_*(\kappa) \subset P_* \subset P_0$ for all $\kappa \geq 0$. More detailed discussions on sufficient and P_0 matrices can be found in the award-winning books of Cottle, Pang, and Stone [11] and Kojima, Megiddo, Noma, and Yoshise [23].

It is well known that $\text{LCP}(M, q)$ can be reformulated as a nonsmooth equation involving a smoothing parameter μ . Instead of solving the original $\text{LCP}(M, q)$, one solves the parameterized equation by a Newton method that iteratively finds a solution of the equation while

gradually reducing μ to zero. This is the so-called smoothing Newton methods or non-interior continuation methods. In the following we simply call them smoothing algorithms. Smoothing algorithms can start from an arbitrary point and generally do not require the intermediate iteration points to be positive. As a result, they are very flexible for numerical implementation. It has been demonstrated that some smoothing algorithms are very efficient [2, 12, 13]. Up to now, a large number of smoothing algorithms have been presented for solving various optimization problems, e.g., [3, 4, 5, 6, 8, 7, 12, 13, 18, 19, 22, 26, 28]. However, most smoothing algorithms require that the solution set of the LCP be nonempty and bounded, a rather strong condition (Often, this condition is not explicitly stated but is implied by some other conditions, see some discussions in [17]). Moreover, a Jacobian non-singularity condition is also required in most smoothing algorithms. Such an assumption implies that the solution set of the problem consists of a single element if the matrix M is a P_0 matrix.

An iteration method (including the smoothing algorithm) usually generates an infinite iteration sequence converging to a solution of the problem. From both the theoretical and practical points of view, however, it is interesting and important to find a solution in a finite number of iterations. In fact, some algorithms in linear programming [24, 30], in the complementarity problem [1, 14, 15, 20, 21, 32], and in the box constrained variational inequality problem [9] have this property. Our objective is to develop a smoothing algorithm for solving the LCP with a sufficient matrix in a finite number of iterations. Compared to other smoothing algorithms with the finite termination property [9, 14, 20, 32], the condition used in our paper is much weaker, namely we only require that \mathcal{S} be nonempty rather than that i) \mathcal{S} be nonempty and bounded and ii) the generalized Jacobian of the smoothing equation be nonsingular on \mathcal{S} and/or the solution be strictly complementary. In our smoothing algorithm, a projection-type testing procedure is included. This testing procedure is a key for us to obtain the finite termination of the algorithm under the weaker condition.

Throughout this paper, we use the following assumption:

Assumption 1.1 *The solution set of $LCP(M, q)$ is nonempty, i.e., $\mathcal{S} \neq \emptyset$.*

We consider $LCP(M, q)$ with M being a sufficient matrix. Under Assumption 1.1 we will show the following results:

- The iteration sequence generated by the proposed algorithm is bounded.

- The algorithm can find a maximally complementary solution to $\text{LCP}(M, q)$ in a finite number of iterations.

To the best of our knowledge, the above convergence properties have not been shown to be simultaneously satisfied by any existing smoothing algorithms.

The paper is organized as follows. In the next section, we introduce a smoothing function and present some preliminary results. In Section 3, we propose a smoothing algorithm and prove some basic results. The convergence properties of the algorithm are discussed in Section 4. We give some final remarks in Section 5.

We assume that all vectors are column vectors. The subscript T denotes the transpose. \mathfrak{R}_+^n (respectively, \mathfrak{R}_{++}^n) denotes the nonnegative (respectively, positive) orthant in \mathfrak{R}^n . As mentioned before, we denote $\mathcal{I} = \{1, 2, \dots, n\}$. For any vector u , we denote by u_i the i th component of u and by u^k the k -th iterate of u . For any subset \mathcal{C} of \mathcal{I} , we use $u_{\mathcal{C}}$ to stand for the vector obtained by removing from u those u_i with $i \notin \mathcal{C}$. We denote by $\|u\|$ the 2-norm of u . For any vectors $u, v \in \mathfrak{R}^n$, we write $(u^T, v^T)^T$ as (u, v) for simplicity and denote by $\min\{u, v\}$ the vector whose i -th component is $\min\{u_i, v_i\}$. We denote by $\mathfrak{R}^{n \times n}$ the space of $n \times n$ real matrices. For any $A \in \mathfrak{R}^{n \times n}$ and $\mathcal{C}, \mathcal{L} \subset \mathcal{I}$, we denote by $A_{\mathcal{C}\mathcal{L}}$ the submatrix of A obtained by removing all rows of A with indices outside of \mathcal{C} and removing all columns of A with indices outside of \mathcal{L} . Also, we denote $\|A\| = \max_{u \in \mathfrak{R}^n, \|u\|=1} \|Au\|$. We denote by $\text{dist}((u, v), \mathcal{S})$ the Euclidean distance of the vector $(u, v) \in \mathfrak{R}^{2n}$ to the solution set \mathcal{S} of $\text{LCP}(M, q)$, i.e.,

$$\text{dist}((u, v), \mathcal{S}) = \inf_{(x, s) \in \mathcal{S}} \|(u, v) - (x, s)\|.$$

E denotes the $n \times n$ identity matrix. While k denotes the iteration index, the set of all iteration indices is written as \mathcal{K} , i.e., $\mathcal{K} := \{0, 1, 2, \dots\}$.

2 The Smoothing Reformulation and Its Properties

It is evident that smoothing functions play a very important role in the smoothing algorithms. Many smoothing functions have been proposed in the literature. In this paper we introduce the following smoothing function:

$$\phi(\mu, a, b) = (1 + \mu)(a + b) - \sqrt{(1 - \mu)^2(a - b)^2 + 4\mu}, \quad (2.2)$$

where $(\mu, a, b) \in \mathfrak{R}^3$. Then it is easy to see that the function ϕ is continuously differentiable at any $(\mu, a, b) \in \mathfrak{R}^3$ with $\mu > 0$. Another fundamental property of the smoothing function

is as follows.

Proposition 2.1 For any $(\mu, a, b, c) \in \Re \times \Re \times \Re \times \Re$, if $\mu \in \Re_{++}$ then

$$\phi(\mu, a, b) = c \iff a + \mu b - \frac{1}{2}c > 0, \mu a + b - \frac{1}{2}c > 0, \text{ and } (a + \mu b - \frac{1}{2}c)(\mu a + b - \frac{1}{2}c) = \mu.$$

Proof. For any $(\mu, a, b, c) \in \Re \times \Re \times \Re \times \Re$ with $\mu \in \Re_{++}$, one has

$$\begin{aligned} \phi(\mu, a, b) = c & \\ \iff (1 + \mu)(a + b) - \sqrt{(1 - \mu)^2(a - b)^2 + 4\mu} - c = 0 & \\ \iff \left(a + \mu b - \frac{1}{2}c\right) + \left(\mu a + b - \frac{1}{2}c\right) = \sqrt{\left[\left(a + \mu b - \frac{1}{2}c\right) - \left(\mu a + b - \frac{1}{2}c\right)\right]^2 + 4\mu} & \\ \iff a + \mu b - \frac{1}{2}c > 0, \mu a + b - \frac{1}{2}c > 0, \text{ and } \left(a + \mu b - \frac{1}{2}c\right) \left(\mu a + b - \frac{1}{2}c\right) = \mu. & \end{aligned}$$

□

For any $(x, s) \in \Re^{2n}$, let

$$H(\mu, x, s) := \begin{bmatrix} \mu \\ F(\mu, x, s) \end{bmatrix}, \quad (2.3)$$

where

$$F(\mu, x, s) := \begin{bmatrix} s - Mx - q \\ \Phi(\mu, x, s) \end{bmatrix} \quad (2.4)$$

with

$$\Phi(\mu, x, s) := \begin{bmatrix} \phi(\mu, x_1, s_1) \\ \vdots \\ \phi(\mu, x_n, s_n) \end{bmatrix}. \quad (2.5)$$

From (2.2), it is easy to see that

$$\phi(0, a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0.$$

Thus, by (2.3), (2.4), and (2.5) we have

$$H(\mu, x, s) = 0 \iff \mu = 0 \text{ and } (x, s) \text{ solves LCP}(M, q). \quad (2.6)$$

This demonstrates that we can solve $\text{LCP}(M, q)$ by using the following approach: reformulate $\text{LCP}(M, q)$ as a system of smooth equations $H(\mu, x, s) = 0$ and then iteratively solve this equation for a root.

The following two propositions describe the basic properties of the smoothing equation $H(\mu, x, s) = 0$. We omit their proofs since the propositions can be obtained in a similar way to [18, Theorems 2.1, 2.2, and Lemma 3.1].

Proposition 2.2 *Suppose that M is a P_0 -matrix. Let*

$$\mathcal{P} := \{(x, s) \in \mathfrak{R}^{2n} : F(\mu, x, s) = 0 \text{ where } \mu > 0\}.$$

Then the following results hold.

- *The path \mathcal{P} exists. The trajectory $(x(\mu), s(\mu)) \in \mathcal{P}$ is continuous in μ on $(0, \infty)$.*
- *Let $\{\mu_k\}$ be a sequence of positive numbers converging to 0 and let $(x(\mu_k), s(\mu_k)) \in \mathcal{P}$. If $(x(\mu_k), s(\mu_k))$ converges to a point (x^*, s^*) , then (x^*, s^*) solves LCP(M, q).*

In the following we use $\text{vec}\{u_i : i \in \mathcal{I}\}$ to denote the vector u , and use $\text{diag}\{u_i : i \in \mathcal{I}\}$ to denote the diagonal matrix whose i -th diagonal element is u_i .

Proposition 2.3 *Let H' denote the Jacobian of the function H defined by (2.3), i.e.,*

$$H'(\mu, x, s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -M & E \\ x + s - d_\mu & (1 + \mu)E - D_x & (1 + \mu)E - D_s \end{bmatrix}, \quad (2.7)$$

where E denotes the $n \times n$ identity matrix, and

$$\begin{aligned} d_\mu &:= \text{vec}\{(d_\mu)_i : i \in \mathcal{I}\}, & \text{where } (d_\mu)_i &= \frac{-(1-\mu)(x_i-s_i)^2+2}{\sqrt{(1-\mu)^2(x_i-s_i)^2+4\mu}} & i \in \mathcal{I}, \\ D_x &:= \text{diag}\{(d_x)_i : i \in \mathcal{I}\}, & \text{where } (d_x)_i &= \frac{(1-\mu)^2(x_i-s_i)}{\sqrt{(1-\mu)^2(x_i-s_i)^2+4\mu}}, & i \in \mathcal{I}, \\ D_s &:= \text{diag}\{(d_s)_i : i \in \mathcal{I}\}, & \text{where } (d_s)_i &= \frac{-(1-\mu)^2(x_i-s_i)}{\sqrt{(1-\mu)^2(x_i-s_i)^2+4\mu}} & i \in \mathcal{I}. \end{aligned}$$

If M is a P_0 -matrix, then H' is non-singular at any point $(\mu, x, s) \in \mathfrak{R}^{1+2n}$ with $\mu > 0$.

3 The Smoothing Algorithm

We will use the following simplified notations

$$z := (\mu, x, s) \quad \text{and} \quad z^k := (\mu_k, x^k, s^k), \quad \forall k \in \mathcal{K}.$$

Algorithm 3.1 *(A Smoothing Algorithm)*

Step 1 Choose $\eta, \delta, \sigma \in (0, 1)$. Let $x^0 \in \mathfrak{R}^n$ be an arbitrary vector and $\mu_0 > 0$ be the initial parameter. Set $s^0 := Mx^0 + q$ and $z^0 := (\mu_0, x^0, s^0)$. Choose $\beta > 1$ such that $\|H(z^0)\| \leq \beta\mu_0$. Set $e^0 := (1, 0, \dots, 0)^T \in \mathfrak{R}^{1+2n}$ and $k := 0$.

Step 2 If $\mu_k > \eta$, then go to Step 3; otherwise, define three sets

$$\begin{aligned} \mathcal{B}^k &:= \{i \in \mathcal{I} : x_i^k > \sqrt[3]{\mu_k} \text{ and } s_i^k \leq \sqrt[3]{\mu_k}\}, \\ \mathcal{N}^k &:= \{i \in \mathcal{I} : x_i^k \leq \sqrt[3]{\mu_k} \text{ and } s_i^k > \sqrt[3]{\mu_k}\}, \\ \mathcal{T}^k &:= \{i \in \mathcal{I} : x_i^k \leq \sqrt[3]{\mu_k} \text{ and } s_i^k \leq \sqrt[3]{\mu_k}\}. \end{aligned} \quad (3.8)$$

If $\mathcal{B}^k \cup \mathcal{N}^k \cup \mathcal{T}^k \neq \mathcal{I}$ or $\mathcal{B}^k \cup \mathcal{N}^k = \emptyset$, then go to Step 3; otherwise, let $(\tilde{x}_{\mathcal{B}^k}, \tilde{s}_{\mathcal{N}^k})$ be the solution of the following projection subproblem

$$\begin{aligned} \min_{x_{\mathcal{B}^k}, s_{\mathcal{N}^k}} & \left\| \begin{pmatrix} x_{\mathcal{B}^k} \\ s_{\mathcal{N}^k} \end{pmatrix} - \begin{pmatrix} x_{\mathcal{B}^k}^k \\ s_{\mathcal{N}^k}^k \end{pmatrix} \right\| \\ \text{subject to} & \begin{bmatrix} -M_{\mathcal{B}^k \mathcal{B}^k} & E_{\mathcal{B}^k \mathcal{N}^k} \\ -M_{\mathcal{N}^k \mathcal{B}^k} & E_{\mathcal{N}^k \mathcal{N}^k} \\ -M_{\mathcal{T}^k \mathcal{B}^k} & E_{\mathcal{T}^k \mathcal{N}^k} \end{bmatrix} \begin{pmatrix} x_{\mathcal{B}^k} \\ s_{\mathcal{N}^k} \end{pmatrix} = \begin{pmatrix} q_{\mathcal{B}^k} \\ q_{\mathcal{N}^k} \\ q_{\mathcal{T}^k} \end{pmatrix}. \end{aligned} \quad (3.9)$$

If $(\tilde{x}_{\mathcal{B}^k}, \tilde{s}_{\mathcal{N}^k}) > 0$, then stop; otherwise go to Step 3.

Step 3 Compute $\Delta z^k := (\Delta\mu_k, \Delta x^k, \Delta s^k) \in \mathfrak{R} \times \mathfrak{R}^n \times \mathfrak{R}^n$ by

$$H(z^k) + H'(z^k)\Delta z^k = (1/\beta)\|H(z^k)\|e^0. \quad (3.10)$$

Step 4 Let λ_k be the maximum of the values $1, \delta, \delta^2, \dots$ such that

$$\|H(z^k + \lambda_k \Delta z^k)\| \leq [1 - \sigma(1 - 1/\beta)\lambda_k]\|H(z^k)\|. \quad (3.11)$$

Step 5 Set $z^{k+1} := z^k + \lambda_k \Delta z^k$ and $k := k + 1$. If $\|H(z^k)\| = 0$, stop; otherwise, go to Step 2.

Remark 3.2

(i) Algorithm 3.1 is a revised version of Algorithm 3.1 in [18] in the sense that if Step 2 is removed from Algorithm 3.1, then the remainder of Algorithm 3.1 is quite similar to the corrector step of Algorithm 3.1 in [18].

(ii) From the first equation of (3.10) it follows that

$$\mu_{k+1} = \mu_k + \lambda_k \Delta\mu_k = (1 - \lambda_k)\mu_k + \lambda_k(1/\beta)\|H(z^k)\| > 0, \quad (3.12)$$

which implies that $\mu_k > 0$ for all $k \in \mathcal{K}$.

(iii) By Proposition 2.3 and the result (ii) above we know that $H'(z^k)$ is non-singular, which implies that Step 3 is well defined. Hence, to show that Algorithm 3.1 is well defined, it suffices to show that the line search (3.11) is well defined. To this end, let

$$r^k(\alpha) := H(z^k + \alpha\Delta z^k) - H(z^k) - \alpha H'(z^k)\Delta z^k.$$

Since H is continuously differentiable at any $z := (\mu, x, s) \in \mathfrak{R}^{1+2n}$ with $\mu > 0$, we have that $\|r^k(\alpha)\| = o(\alpha)$. Then by (3.10), we obtain

$$\begin{aligned} \|H(z^k + \alpha\Delta z^k)\| &= \|r^k(\alpha) + H(z^k) + \alpha H'(z^k)\Delta z^k\| \\ &\leq \|r^k(\alpha)\| + [1 - \alpha(1 - 1/\beta)]\|H(z^k)\| \\ &= o(\alpha) + [1 - \alpha(1 - 1/\beta)]\|H(z^k)\|, \end{aligned} \quad (3.13)$$

which shows that the line search (3.11) is well defined.

(iv) Step 2 is a projection-type testing procedure. From Algorithm 3.1, it is not difficult to see that if Algorithm 3.1 terminates at Step 2 for some iteration index $k_0 > 0$, then

$$(\tilde{x}, \tilde{s}) := \left((\tilde{x}_{\mathcal{B}^{k_0}}^{k_0}, 0, 0), (0, \tilde{s}_{\mathcal{N}^{k_0}}^{k_0}, 0) \right) \in \mathfrak{R}^n \times \mathfrak{R}^n$$

is a solution to LCP (M, q) .

The following lemma describes some properties of the iteration sequence generated by Algorithm 3.1, which will be used in our analysis later.

Lemma 3.3 *Let the sequence $\{z^k\}$ be generated by Algorithm 3.1. Then the following results hold.*

- (i) *The sequence $\{\|H(z^k)\|\}$ is monotonically decreasing.*
- (ii) *$z^k \in \Omega := \{z \in \mathfrak{R}_+ \times \mathfrak{R}^n \times \mathfrak{R}^n : (x, s) \in \mathcal{F} \text{ and } \|H(z)\| \leq \beta\mu\}$ for all $k \in \mathcal{K}$.*
- (iii) *The sequence $\{\mu_k\}$ is monotonically decreasing.*

Proof.

(i) By (3.11) the result is obvious.

(ii) From the choice of the initial point we know that $z^0 \in \Omega$. Assume that $z^k \in \Omega$, we will prove that $z^{k+1} \in \Omega$. Combining (3.10) and the fact that $(x^k, s^k) \in \mathcal{F}$, it follows that

$(x^{k+1}, s^{k+1}) \in \mathcal{F}$. On the other hand, we have

$$\begin{aligned} \mu^{k+1} - (1/\beta)\|H(z^{k+1})\| &= (1 - \lambda_k)\mu^k + \lambda_k(1/\beta)\|H(z^k)\| - (1/\beta)\|H(z^{k+1})\| \\ &\geq (1/\beta)(\|H(z^k)\| - \|H(z^{k+1})\|) \\ &\geq 0, \end{aligned}$$

where the equality follows from (3.12); the first inequality from the assumption, i.e., $\mu^k \geq (1/\beta)\|H(z^k)\|$; and the last inequality from the above result (i).

(iii) Combining (3.12) and the above result (ii), we have

$$\mu^{k+1} = (1 - \lambda_k)\mu^k + \lambda_k(1/\beta)\|H(z^k)\| \leq (1 - \lambda_k)\mu^k + \lambda_k\mu^k = \mu^k,$$

which implies that $\{\mu_k\}$ is monotonically decreasing. The proof is completed. \square

4 Convergence of Algorithm 3.1

We will show that Algorithm 3.1 can find a maximally complementary solution of LCP(M, q) in a finite number of iterations. However, since the analysis of maximal complementarity and the finite convergence heavily depend on the asymptotic property of the iterates, we divided the analysis into three subsections. In the first two subsections, we assume that Step 2 of the algorithm is skipped at each iteration, so the algorithm will generate an infinite sequence. Equipped with the properties of this “nonstop” algorithm, we then proceed to show the maximal complementarity and the finite convergence of Algorithm 3.1 in the third subsection.

4.1 Global Convergence of the Nonstop Algorithm

Assuming that Step 2 of the algorithm is skipped at each iteration, which is from now on called the *nonstop algorithm*, we first investigate the boundedness of iteration sequence generated by such an algorithm.

Lemma 4.1 *Suppose that M is a sufficient matrix. Let the sequence $\{z^k\}$ be generated by the nonstop algorithm. If Assumption 1.1 holds, then the iteration sequence $\{z^k\}$ is bounded.*

Proof. Since Assumption 1.1 holds, there exists a point $(x^*, s^*) \in \mathcal{S}$.

Firstly, by Lemma 3.3(iii) we obtain that the sequence $\{\mu_k\}$ is bounded.

Secondly, we prove that the sequence $\{x^k\}$ is bounded. Assume that the sequence $\{x^k\}$ is unbounded, we will derive a contradiction. By using Lemma 3.3(ii) we have

$$s^k = Mx^k + q, \quad \forall k \in \mathcal{K} \quad (4.14)$$

and

$$\|H(z^k)\| \leq \beta\mu_k, \quad \forall k \in \mathcal{K}. \quad (4.15)$$

Thus, by (2.3), (4.14), and (4.15) we have

$$\|\Phi(z^k)\| \leq \sqrt{\beta^2 - 1}\mu_k, \quad \forall k \in \mathcal{K}. \quad (4.16)$$

Construct a sequence $\{\Gamma^k\}$ by

$$\Gamma^k := \frac{1}{\mu_k}\Phi(\mu_k, x^k, s^k), \quad \forall k \in \mathcal{K}. \quad (4.17)$$

Then (4.16) and (4.17) imply that $\|\Gamma^k\| \leq \sqrt{\beta^2 - 1}$ for all $k \in \mathcal{K}$, so the sequence $\{\Gamma^k\}$ is uniformly bounded. Now, we construct another sequence $\{(\hat{x}^k, \hat{s}^k)\}$ by

$$\hat{x}^k := x^k + \mu_k s^k - \frac{1}{2}\mu_k \Gamma^k \quad \text{and} \quad \hat{s}^k := s^k + \mu_k x^k - \frac{1}{2}\mu_k \Gamma^k, \quad \forall k \in \mathcal{K}. \quad (4.18)$$

Then, by using Proposition 2.1, we have

$$\hat{x}_i^k \geq 0, \quad \hat{s}_i^k \geq 0, \quad \text{and} \quad \hat{x}_i^k \hat{s}_i^k = \mu_k, \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}. \quad (4.19)$$

Since, for any $k \in \mathcal{K}$,

$$\begin{aligned} \hat{x}^k - x^* &= x^k - x^* + \mu_k s^k - \frac{1}{2}\mu_k \Gamma^k, \\ \hat{s}^k - s^* &= s^k - s^* + \mu_k x^k - \frac{1}{2}\mu_k \Gamma^k = M(x^k - x^*) + \mu_k x^k - \frac{1}{2}\mu_k \Gamma^k, \end{aligned}$$

we have

$$\begin{aligned} & (\hat{x}^k - x^*)^T (\hat{s}^k - s^*) \\ &= (x^k - x^*)^T M(x^k - x^*) + (x^k - x^*)^T (\mu_k x^k - \frac{1}{2}\mu_k \Gamma^k) \\ & \quad + (\mu_k(Mx^k + q) - \frac{1}{2}\mu_k \Gamma^k)^T (M(x^k - x^*) + \mu_k x^k - \frac{1}{2}\mu_k \Gamma^k) \\ &= (x^k - x^*)^T M(x^k - x^*) + \mu_k \|Mx^k\|^2 + \mu_k^2 (x^k)^T Mx^k + \mu_k g(\mu_k, x^k) \\ &\geq -4\kappa \sum_{i \in I_+(x^k - x^*)} (x_i^k - x_i^*) (Mx^k - Mx^*)_i - 4\kappa \mu_k^2 \sum_{i \in I_+(x^k)} x_i^k (Mx^k)_i + \mu_k g(\mu_k, x^k) \\ &= -4\kappa \sum_{i \in I_+(x^k - x^*)} (x_i^k - x_i^*) (s_i^k - s_i^*) - 4\kappa \mu_k^2 \sum_{i \in I_+(x^k)} (x_i^k s_i^k - q_i x_i^k) + \mu_k g(\mu_k, x^k), \quad (4.20) \end{aligned}$$

where the inequality follows from (4.1) and the function g is defined by

$$\begin{aligned} g(\mu_k, x^k) &:= (x^k - x^*)^T(x^k - \frac{1}{2}\Gamma^k) + (Mx^k)^T(-Mx^* - \frac{1}{2}\Gamma^k) \\ &\quad + (q - \frac{1}{2}\Gamma^k)^T(M(x^k - x^*) + \mu_k x^k - \frac{1}{2}\mu_k \Gamma^k). \end{aligned} \quad (4.21)$$

By (4.18) we obtain that, for any $i \in \mathcal{I}$ and any $k \in \mathcal{K}$,

$$\begin{aligned} \hat{x}_i^k \hat{s}_i^k &= (x_i^k + \mu_k s_i^k - \frac{1}{2}\mu_k \Gamma_i^k)(\mu_k x_i^k + s_i^k - \frac{1}{2}\mu_k \Gamma_i^k) \\ &= \mu_k (x_i^k)^2 + (1 + \mu_k^2)x_i^k s_i^k + \mu_k (s_i^k)^2 - \frac{1}{2}(1 + \mu_k)\mu_k \Gamma_i^k (x_i^k + s_i^k) + \frac{1}{4}\mu_k^2 (\Gamma_i^k)^2 \\ &\geq (1 + \mu_k^2)x_i^k s_i^k - \frac{1}{2}(1 + \mu_k)\mu_k \Gamma_i^k (x_i^k + (Mx^k + q)_i), \end{aligned}$$

i.e.,

$$\begin{aligned} x_i^k s_i^k &\leq \frac{\hat{x}_i^k \hat{s}_i^k}{1 + \mu_k^2} + \frac{(1 + \mu_k)\mu_k}{2(1 + \mu_k^2)} \Gamma_i^k (x_i^k + (Mx^k + q)_i) \\ &= \frac{\mu_k}{1 + \mu_k^2} + \frac{(1 + \mu_k)\mu_k}{2(1 + \mu_k^2)} \Gamma_i^k (x_i^k + (Mx^k + q)_i); \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} &(\hat{x}_i^k - x_i^k)(\hat{s}_i^k - s_i^*) \\ &= (x_i^k - x_i^* + \mu_k s_i^k - \frac{1}{2}\mu_k \Gamma_i^k)(s_i^k - s_i^* + \mu_k x_i^k - \frac{1}{2}\mu_k \Gamma_i^k) \\ &= (x_i^k - x_i^*)(s_i^k - s_i^*) + (x_i^k - x_i^*)(\mu_k x_i^k - \frac{1}{2}\mu_k \Gamma_i^k) \\ &\quad + (\mu_k s_i^k - \frac{1}{2}\mu_k \Gamma_i^k)(s_i^k - s_i^*) + (\mu_k s_i^k - \frac{1}{2}\mu_k \Gamma_i^k)(\mu_k x_i^k - \frac{1}{2}\mu_k \Gamma_i^k) \\ &= (x_i^k - x_i^*)(s_i^k - s_i^*) + (x_i^k - x_i^*)(\mu_k x_i^k - \frac{1}{2}\mu_k \Gamma_i^k) \\ &\quad + \mu_k (s_i^k)^2 - \mu_k (Mx^k + q)_i s_i^* - \frac{1}{2}\mu_k \Gamma_i^k (Mx^k + q)_i - \frac{1}{2}\mu_k \Gamma_i^k s_i^* \\ &\quad + \mu_k^2 x_i^k s_i^k - \frac{1}{2}\mu_k^2 \Gamma_i^k (x_i^k + s_i^k) + \frac{1}{4}\mu_k^2 (\Gamma_i^k)^2 \\ &\geq (x_i^k - x_i^*)(s_i^k - s_i^*) + (x_i^k - x_i^*)(\mu_k x_i^k - \frac{1}{2}\mu_k \Gamma_i^k) \\ &\quad - \mu_k (Mx^k + q)_i s_i^* - \frac{1}{2}\mu_k \Gamma_i^k (Mx^k + q)_i - \frac{1}{2}\mu_k \Gamma_i^k s_i^* \\ &\quad + \mu_k^2 x_i^k s_i^k - \frac{1}{2}\mu_k^2 \Gamma_i^k (x_i^k + s_i^k). \end{aligned} \quad (4.23)$$

For any $i \in \mathcal{I}$ and any $k \in \mathcal{K}$, noting that

$$x_i^k s_i^k = (x_i^k - x_i^*)(s_i^k - s_i^*) + x_i^k s_i^* + x_i^* (Mx^k + q)_i, \quad (4.24)$$

and letting

$$\begin{aligned}
f_i(\mu_k, x^k) &:= (x_i^k - x_i^*)(x_i^k - \frac{1}{2}\Gamma_i^k) - (Mx^k + q)_i s_i^* - \frac{1}{2}\Gamma_i^k (Mx^k + q)_i - \frac{1}{2}\Gamma_i^k s_i^* \\
&\quad + \mu_k x_i^k s_i^* + \mu_k x_i^* (Mx^k + q)_i - \frac{1}{2}\mu_k \Gamma_i^k (Mx^k + q)_i,
\end{aligned} \tag{4.25}$$

it follows from (4.23)-(4.25) and (4.19) that

$$\begin{aligned}
(x_i^k - x_i^*)(s_i^k - s_i^*) &\leq \frac{1}{1 + \mu_k^2} (\hat{x}_i^k - x_i^*)(\hat{s}_i^k - s_i^*) - \frac{\mu_k}{1 + \mu_k^2} f_i(\mu_k, x^k) \\
&= \frac{1}{1 + \mu_k^2} (\hat{x}_i^k \hat{s}_i^k - \hat{x}_i^k s_i^* - x_i^* \hat{s}_i^k) - \frac{\mu_k}{1 + \mu_k^2} f_i(\mu_k, x^k) \\
&\leq \frac{\mu_k}{1 + \mu_k^2} - \frac{\mu_k}{1 + \mu_k^2} f_i(\mu_k, x^k)
\end{aligned} \tag{4.26}$$

By combining (4.20) with (4.22) and (4.26), we obtain that, for any $i \in \mathcal{I}$ and any $k \in \mathcal{K}$,

$$\begin{aligned}
& -(\hat{x}^k - x^*)^T (\hat{s}^k - s^*) \\
\leq & 4\kappa \sum_{i \in I_+(x^k - x^*)} \left(\frac{\mu_k}{1 + \mu_k^2} - \frac{\mu_k}{1 + \mu_k^2} f_i(\mu_k, x^k) \right) \\
& + 4\kappa \mu_k^2 \sum_{i \in I_+(x^k)} (x_i^k s_i^k - q_i x_i^k) - \mu_k g(\mu_k, x^k) \\
\leq & 4\kappa \sum_{i \in I_+(x^k - x^*)} \left(\frac{\mu_k}{1 + \mu_k^2} - \frac{\mu_k}{1 + \mu_k^2} f_i(\mu_k, x^k) \right) - \mu_k g(\mu_k, x^k) \\
& + 4\kappa \mu_k^2 \sum_{i \in I_+(x^k)} \left(\frac{\mu_k}{1 + \mu_k^2} + \frac{(1 + \mu_k)\mu_k}{2(1 + \mu_k^2)} \Gamma_i^k (x_i^k + (Mx^k + q)_i) - q_i x_i^k \right).
\end{aligned}$$

Since, for any $k \in \mathcal{K}$,

$$\begin{aligned}
0 \leq (\hat{x}^k)^T s^* + (x^*)^T \hat{s}^k &= -(\hat{x}^k - x^*)^T (\hat{s}^k - s^*) + (\hat{x}^k)^T \hat{s}^k \\
&= -(\hat{x}^k - x^*)^T (\hat{s}^k - s^*) + n\mu_k,
\end{aligned}$$

we further obtain that

$$\begin{aligned}
& \mu_k g(\mu_k, x^k) + \frac{4\kappa\mu_k}{1 + \mu_k^2} \sum_{i \in I_+(x^k - x^*)} f_i(\mu_k, x^k) \\
& \quad - 4\kappa\mu_k^2 \sum_{i \in I_+(x^k)} \left(\frac{(1 + \mu_k)\mu_k}{2(1 + \mu_k^2)} \Gamma_i^k (x_i^k + (Mx^k + q)_i) - q_i x_i^k \right) \\
\leq & 4\kappa \sum_{i \in I_+(x^k - x^*)} \frac{\mu_k}{1 + \mu_k^2} + 4\kappa\mu_k^2 \sum_{i \in I_+(x^k)} \frac{\mu_k}{1 + \mu_k^2} + n\mu_k \\
\leq & \frac{4\kappa n\mu_k}{1 + \mu_k^2} + \frac{4\kappa n\mu_k^3}{1 + \mu_k^2} + n = (4\kappa n + n)\mu_k,
\end{aligned}$$

i.e.,

$$\begin{aligned}
& g(\mu_k, x^k) + \frac{4\kappa}{1 + \mu_k^2} \sum_{i \in I_+(x^k - x^*)} f_i(\mu_k, x^k) \\
& \quad - 4\kappa\mu_k \sum_{i \in I_+(x^k)} \left(\frac{(1 + \mu_k)\mu_k}{2(1 + \mu_k^2)} \Gamma_i^k(x_i^k + (Mx^k + q)_i) - q_i x_i^k \right) \\
& \leq 4\kappa n + n.
\end{aligned} \tag{4.27}$$

Now, we construct two index sets \mathcal{J}^∞ and $\mathcal{J}_{\max}^\infty$ by

$$\begin{aligned}
\mathcal{J}^\infty & := \{i \in \mathcal{I} : |x_i^k| \rightarrow \infty \text{ as } k \rightarrow \infty\}, \\
\mathcal{J}_{\max}^\infty & := \left\{ i \in \mathcal{J}^\infty : \begin{array}{l} \text{for each } j \in \mathcal{J}^\infty, \lim_{k \rightarrow \infty} \frac{|x_j^k|}{|x_i^k|} = c_j \text{ where} \\ c_j \text{ is either a positive scalar or } +\infty \end{array} \right\}.
\end{aligned}$$

Since the sequence $\{x^k\}$ is assumed to be unbounded, it follows that $\mathcal{J}^\infty \neq \emptyset$ and $\mathcal{J}_{\max}^\infty \neq \emptyset$. By using the definition (4.21) of the function g , we have

$$g(\mu_k, x^k) = O((x_{i_0}^k)^2), \quad \forall i_0 \in \mathcal{J}_{\max}^\infty, k \in \mathcal{K}. \tag{4.28}$$

If $\mathcal{J}^\infty \cap I_+(x^k - x^*) \neq \emptyset$ for all $k \in \mathcal{K}$, then there exist a subsequence of $\{z^k\}$, which is denoted by $\{z^k\}_{k \in \hat{\mathcal{K}}}$, and at least an index $i_0 \in \mathcal{I}$ such that $i_0 \in \mathcal{J}^\infty \cap I_+(x^k - x^*)$ for all $k \in \hat{\mathcal{K}}$. Furthermore, by definitions (4.25) of the functions f_i ($i \in \mathcal{I}$), we have

$$f_{i_0}(\mu_k, x^k) = O((x_{i_0}^k)^2), \quad \forall k \in \hat{\mathcal{K}}. \tag{4.29}$$

Similarly, if $\mathcal{J}^\infty \cap I_+(x^k) \neq \emptyset$, then there exist a subsequence of $\{z^k\}$, which is denoted by $\{z^k\}_{k \in \tilde{\mathcal{K}}}$, and at least an index $i_0 \in \mathcal{I}$ such that $i_0 \in \mathcal{J}^\infty \cap I_+(x^k)$ for all $k \in \tilde{\mathcal{K}}$. Furthermore,

$$\left| \frac{(1 + \mu_k)\mu_k}{2(1 + \mu_k^2)} \Gamma_{i_0}^k(x_{i_0}^k + (Mx^k + q)_{i_0}) - q_{i_0} x_{i_0}^k \right| = O(|x_{i_0}^k|), \quad \forall k \in \tilde{\mathcal{K}}. \tag{4.30}$$

Combining (4.28) with (4.29) and (4.30), we obtain that there exists at least a subsequence of $\{z^k\}$ such that the left side of (4.27) always tends to $+\infty$ along the subsequence, whereas the right side of (4.27) is bounded by a constant, a contradiction. Thus, the sequence $\{x^k\}$ is bounded.

The result that $\{s^k\}$ is bounded can be easily obtained from (4.14) and the boundedness of $\{x^k\}$. The proof is completed. \square

To assure the boundedness of the iteration sequence generated by some smoothing algorithm, various conditions have been proposed in the literature. However, the conditions

used in most smoothing algorithms imply that the solution set of $\text{LCP}(M, q)$ is nonempty and bounded (see, for example, [17] for some discussions). Thus, the assumptions used in Lemma 4.1 is weaker than those required by most smoothing algorithms. It should be pointed out that Assumption 1.1 has been used in regularized non-interior continuation algorithms [31]. In addition, some related condition discussed in [10] also allows the unboundedness of the solution set of the underlying LCP.

The following global convergence property of the nonstop algorithm can be obtained in a similar way to other smoothing algorithms, e.g., [18, Theorem 3.1].

Theorem 4.2 *Suppose that M is a sufficient matrix. If Assumption 1.1 is satisfied, then we have either*

- (i) *Algorithm 3.1 terminates at Step 2 (and hence a solution of $\text{LCP}(M, q)$ is obtained),*
or
- (ii) *Algorithm 3.1 generates an infinite sequence $\{z^k\}$ which satisfies that*
 - (a) *$\lim_{k \rightarrow \infty} \|H(z^k)\| = 0$ and $\lim_{k \rightarrow \infty} \mu_k = 0$; and that*
 - (b) *every accumulation point of the sequence $\{(x^k, s^k)\}$ is a solution of $\text{LCP}(M, q)$.*

4.2 Maximal Complementarity of the Nonstop Algorithm

In the previous subsection, we proved that a solution of $\text{LCP}(M, q)$ can be found by the nonstop algorithm. In this subsection, we show that this solution is a maximally complementary solution of $\text{LCP}(M, q)$. Recall that a solution (x^*, s^*) of $\text{LCP}(M, q)$ is said to be a maximally complementary solution if the number of its positive components is maximal. Note that the indices of those positive components are invariant among all maximally complementary solutions of $\text{LCP}(M, q)$ when M is a sufficient matrix. Let

$$\begin{aligned}
 \mathcal{B} &:= \{i \in \mathcal{I} : x_i^* > 0 \text{ and } s_i^* = 0\}, \\
 \mathcal{N} &:= \{i \in \mathcal{I} : x_i^* = 0 \text{ and } s_i^* > 0\}, \\
 \mathcal{T} &:= \{i \in \mathcal{I} : x_i^* = s_i^* = 0\}.
 \end{aligned} \tag{4.31}$$

Then \mathcal{B}, \mathcal{N} , and \mathcal{T} form a partition of index set \mathcal{I} , independent of the choice of (x^*, s^*) .

Theorem 4.3 *Suppose that M is a sufficient matrix and that Assumption 1.1 is satisfied. Then the solution produced by the nonstop algorithm is a maximally complementary solution of $\text{LCP}(M, q)$.*

Proof. By Theorem 4.2 we know that every accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$. By Lemma 4.1 we know that $\{z^k\}$ is bounded. Without loss of generality, we assume that $\tilde{z} := (\tilde{\mu}, \tilde{x}, \tilde{s}) \in \mathfrak{R}^{1+2n}$ is an accumulation point of the iteration sequence $\{z^k\}_{k \in \mathcal{K}}$ and $\{z^k\}_{k \in \bar{\mathcal{K}} \subseteq \mathcal{K}}$ is a subsequence converging to \tilde{z} . Then, $\tilde{\mu} = 0$ and (\tilde{x}, \tilde{s}) is a solution of LCP(M, q).

The notations used in the following are the same as those used in Lemma 4.1 except that $k \in \mathcal{K}$ is replaced by $k \in \bar{\mathcal{K}}$. From the proof of (4.20) we have that, for all $k \in \bar{\mathcal{K}}$,

$$\begin{aligned} & (\hat{x}^k - x^*)^T (\hat{s}^k - s^*) \\ &= (x^k - x^*)^T M(x^k - x^*) + \mu_k \|Mx^k\|^2 + \mu_k^2 (x^k)^T Mx^k + \mu_k g(\mu_k, x^k) \\ &\geq -4\kappa \sum_{i \in I_+(x^k - x^*)} (x_i^k - x_i^*) (s_i^k - s_i^*) + \mu_k^2 (x^k)^T Mx^k + \mu_k g(\mu_k, x^k), \end{aligned}$$

where the function g is defined by (4.21). Thus by using (4.26) we obtain that, for all $k \in \bar{\mathcal{K}}$,

$$\begin{aligned} (\hat{x}^k)^T s^* + (x^*)^T \hat{s}^k &= -(\hat{x}^k - x^*)^T (\hat{s}^k - s^*) + n\mu_k \\ &\leq 4\kappa \sum_{i \in I_+(x^k - x^*)} \left(\frac{\mu_k}{1 + \mu_k^2} - \frac{\mu_k}{1 + \mu_k^2} f_i(\mu_k, x^k) \right) \\ &\quad + \mu_k^2 (x^k)^T Mx^k + \mu_k g(\mu_k, x^k) + n\mu_k \\ &\leq \mu_k h(\mu_k, x^k), \end{aligned}$$

where

$$h(\mu_k, x^k) := \frac{4\kappa}{1 + \mu_k^2} \sum_{i \in I_+(x^k - x^*)} \left(1 + |f_i(\mu_k, x^k)| \right) + \mu_k |(x^k)^T Mx^k| + |g(\mu_k, x^k)| + n \quad (4.32)$$

with the functions f_i ($i \in \mathcal{I}$) being defined by (4.25). Using the definitions of \mathcal{B} and \mathcal{N} (see (4.31)), we further obtain that

$$(\hat{x}_{\mathcal{N}}^k)^T s_{\mathcal{N}}^* + (x_{\mathcal{B}}^*)^T \hat{s}_{\mathcal{B}}^k \leq \mu_k h(\mu_k, x^k), \quad \forall k \in \bar{\mathcal{K}}. \quad (4.33)$$

Since

$$\hat{x}_i^k > 0, \quad \hat{s}_i^k > 0, \quad \text{and} \quad \hat{x}_i^k \hat{s}_i^k = \mu_k, \quad \forall i \in \mathcal{I}, \forall k \in \bar{\mathcal{K}},$$

it follows from (4.33) that, for all $k \in \bar{\mathcal{K}}$,

$$\hat{x}_i^k \geq \frac{x_i^*}{h(\mu_k, x^k)} \quad \forall i \in \mathcal{B} \quad \text{and} \quad \hat{s}_i^k \geq \frac{s_i^*}{h(\mu_k, x^k)} \quad \forall i \in \mathcal{N}. \quad (4.34)$$

Noticing that $\lim_{\bar{\mathcal{K}} \ni k \rightarrow \infty} z^k = \tilde{z}$ with $\tilde{\mu} = 0$, we have the following two results:

(i) From (4.18) it follows that

$$\lim_{k \in \bar{\mathcal{K}}} \hat{x}^k = \lim_{k \in \bar{\mathcal{K}}} x^k = \tilde{x} \quad \text{and} \quad \lim_{k \in \bar{\mathcal{K}}} \hat{s}^k = \lim_{k \in \bar{\mathcal{K}}} s^k = \tilde{s}.$$

(ii) By the definitions of g and f_i for any $i \in \mathcal{I}$ (see (4.21) and (4.25)), we know that, for all $k \in \bar{\mathcal{K}}$, the sequences $\{g(\mu_k, x^k)\}$ and $\{f_i(\mu_k, x^k)\}$ have limit points; and hence by the definition of h (see (4.32)) we have the limit of $\{h(\mu_k, x^k)\}$ exists, which is denoted by $h(\tilde{\mu}, \tilde{x})$. It is easy to see that

$$h(\tilde{\mu}, \tilde{x}) > 0.$$

Therefore, by letting $\bar{\mathcal{K}} \ni k \rightarrow \infty$, (4.34) leads to

$$\tilde{x}_i \geq \frac{x_i^*}{h(\tilde{\mu}, \tilde{x})} > 0 \quad \forall i \in \mathcal{B} \quad \text{and} \quad \tilde{s}_i \geq \frac{s_i^*}{h(\tilde{\mu}, \tilde{x})} > 0 \quad \forall i \in \mathcal{N}.$$

These and the fact that (\tilde{x}, \tilde{s}) is a solution of $\text{LCP}(M, q)$ imply that (\tilde{x}, \tilde{s}) is a maximally complementary solution of $\text{LCP}(M, q)$. The proof is completed. \square

4.3 Maximal Complementarity and Finite Termination

In this subsection, we return to Algorithm 3.1. Note that if in an iteration Algorithm 3.1 does not stop at Step 2, then the next iterate is exactly the same as produced by the nonstop algorithm.

We first establish a local error bound result, where the distance from the iteration point to the solution set \mathcal{S} can be bounded by the sequence of smoothing parameters. The established error bound result will help to identify the partition of index set \mathcal{I} of the iteration sequence by giving the upper (or lower) bounds of each component of the iteration points. As a result, we establish that if Algorithm 3.1 terminates at Step 2 at sufficiently large k , then the obtained solution is a maximally complementary solution of $\text{LCP}(M, q)$.

We need the following proposition which can be easily obtained by using a local upper Lipschitz property of a polyhedral multifunction given by Robinson [27] (also see [25]).

Proposition 4.4 *There are two constants $\varepsilon > 0$ and $c > 0$ such that*

$$\text{dist}((x, s), \mathcal{S}) \leq c \|\min\{x, s\}\|$$

for any $(x, s) \in \mathcal{F}$ satisfying $\|\min\{x, s\}\| \leq \varepsilon$.

Lemma 4.5 *Let M be a sufficient matrix and Assumption 1.1 be satisfied. Suppose that the infinite sequence $\{z^k\}$ is generated by the nonstop algorithm. Then there is a constant $c_1 > 0$ such that*

$$\text{dist}((x^k, s^k), \mathcal{S}) \leq c_1 \sqrt{\mu_k}$$

for all sufficiently large $k \in \mathcal{K}$.

Proof. By Lemma 3.3(ii) we have that $(x^k, s^k) \in \mathcal{F}$ for all $k \in \mathcal{K}$, and by Theorem 4.3 and Proposition 2.1 we have that $\min\{x^k, s^k\} \rightarrow 0$ as $k \rightarrow \infty$. Thus, from Proposition 4.4 we know that there is a constant $c_2 > 0$ such that

$$\text{dist}((x^k, s^k), \mathcal{S}) \leq c_2 \|\min\{x^k, s^k\}\| \quad (4.35)$$

holds for all sufficiently large $k \in \mathcal{K}$.

For any $(\mu, a, b) \in \mathfrak{R}_+ \times \mathfrak{R} \times \mathfrak{R}$, from the definition of ϕ we have

$$\begin{aligned} |2 \min\{a, b\} - \phi(\mu, a, b)| &= \left| \sqrt{(1-\mu)^2(a-b)^2 + 4\mu} - \sqrt{(a-b)^2} - \mu(a+b) \right| \\ &= \left| \frac{(a-b)^2 - (1-\mu)^2(a-b)^2 - 4\mu}{\sqrt{(1-\mu)^2(a-b)^2 + 4\mu} + \sqrt{(a-b)^2}} + \mu(a+b) \right| \\ &\leq \frac{|(2\mu - \mu^2)(a-b)^2 - 4\mu|}{2\sqrt{\mu}} + \mu|a+b|. \end{aligned}$$

If (μ, a, b) is bounded, then the above inequality implies that there is a constant $c_3 > 0$ such that

$$|2 \min\{a, b\} - \phi(\mu, a, b)| \leq c_3 \sqrt{\mu},$$

which indicates that there are constants $\alpha_i > 0$ such that

$$|2 \min\{x_i^k, s_i^k\}| \leq |\phi(\mu_k, x_i^k, s_i^k)| + \alpha_i \sqrt{\mu_k}$$

holds for all $i \in \mathcal{I}$ and $k \in \mathcal{K}$. This together with the definition of H yields

$$\|2 \min\{x^k, s^k\}\| \leq \sqrt{\sum_{i=1}^n (|\phi(\mu_k, x_i^k, s_i^k)| + \alpha_i \sqrt{\mu_k})^2} \leq \sqrt{\sum_{i=1}^n (\|H(z^k)\| + \alpha_i \sqrt{\mu_k})^2} \quad (4.36)$$

for all $k \in \mathcal{K}$. By Lemma 3.3(ii) we have

$$\|H(z^k)\| \leq \beta \mu_k, \quad \forall k \in \mathcal{K}. \quad (4.37)$$

Thus it follows from (4.35), (4.36), and (4.37) that there is a constant $c_1 > 0$ such that

$$\text{dist}((x^k, s^k), \mathcal{S}) \leq c_1 \sqrt{\mu_k}$$

for all sufficiently large k . This completes the proof. \square

Suppose that Algorithm 3.1 does not terminate at Step 2. Let (μ_*, x^*, s^*) be an accumulation point of the iteration sequence $\{z^k\}$ generated by the algorithm. Then by Theorems 4.2 and 4.3 we know that $\mu_* = 0$ and (x^*, s^*) is a maximally complementary solution to $\text{LCP}(M, q)$. Thus,

$$x_{\mathcal{B}}^* > 0, \quad s_{\mathcal{N}}^* > 0, \quad \text{and} \quad x_{\mathcal{N}}^* = s_{\mathcal{B}}^* = x_{\mathcal{T}}^* = s_{\mathcal{T}}^* = 0.$$

Let $\{z^k\}_{k \in \bar{\mathcal{K}} \subseteq \mathcal{K}}$ be a convergent subsequence of the iteration sequence and $z^* := (0, x^*, s^*)$ be the limit point of the subsequence. For any $k \in \bar{\mathcal{K}}$, let (x^{k*}, s^{k*}) be the projection of (x^k, s^k) onto the solution set \mathcal{S} . Then by Lemma 4.5 we have

$$\|(x^k, s^k) - (x^{k*}, s^{k*})\| \leq c_1 \sqrt{\mu_k} \quad (4.38)$$

for all sufficiently large $k \in \bar{\mathcal{K}}$. Since $(x^k, s^k) \rightarrow (x^*, s^*)$ and $\mu_k \rightarrow 0$ as $k \rightarrow \infty$, it follows that

$$\|(x^{k*}, s^{k*}) - (x^*, s^*)\| \leq \|(x^{k*}, s^{k*}) - (x^k, s^k)\| + \|(x^k, s^k) - (x^*, s^*)\| \rightarrow 0$$

as $k \rightarrow \infty$. Thus, for all sufficiently large $k \in \bar{\mathcal{K}}$,

$$x_{\mathcal{B}}^{k*} > 0, \quad s_{\mathcal{N}}^{k*} > 0, \quad \text{and} \quad x_{\mathcal{N}}^{k*} = s_{\mathcal{B}}^{k*} = x_{\mathcal{T}}^{k*} = s_{\mathcal{T}}^{k*} = 0. \quad (4.39)$$

Lemma 4.6 *Let M be a sufficient matrix and Assumption 1.1 be satisfied. Suppose that $(0, x^*, s^*)$ is an accumulation point of the infinite sequence $\{z^k\}_{k \in \mathcal{K}}$ generated by the non-stop algorithm and $\{z^k\}_{k \in \bar{\mathcal{K}} \subseteq \mathcal{K}}$ is a subsequence which converges to $z^* := (0, x^*, s^*)$. Let (x^{k*}, s^{k*}) be the projection of (x^k, s^k) on the solution set \mathcal{S} for each $k \in \bar{\mathcal{K}}$. Then, for all sufficiently large $k \in \bar{\mathcal{K}}$ and any $i \in \mathcal{I}$,*

$$\begin{aligned} x_i^k > \sqrt[3]{\mu_k} &\iff x_i^{k*} > 0 & x_i^k \leq \sqrt[3]{\mu_k} &\iff x_i^{k*} = 0 \\ s_i^k > \sqrt[3]{\mu_k} &\iff s_i^{k*} > 0 & s_i^k \leq \sqrt[3]{\mu_k} &\iff s_i^{k*} = 0. \end{aligned}$$

Proof. We only show that $x_i^k > \sqrt[3]{\mu_k} \iff x_i^{k*} > 0$. The other cases can be proved similarly.

If $x_i^k > \sqrt[3]{\mu_k}$ with $k \in \bar{\mathcal{K}}$ being sufficiently large, then by (4.38) we have

$$|x_i^k - x_i^{k*}| \leq \|(x^k, s^k) - (x^{k*}, s^{k*})\| \leq c_1 \sqrt{\mu_k}, \quad (4.40)$$

which implies

$$x_i^{k*} \geq x_i^k - c_1 \sqrt{\mu_k} > \sqrt[3]{\mu_k} - c_1 \sqrt{\mu_k}.$$

Since $\mu_k \rightarrow 0$ as $k \rightarrow \infty$, it follows that $\sqrt[3]{\mu_k} > c_1 \sqrt{\mu_k}$ when $k \in \bar{\mathcal{K}}$ is sufficiently large. Thus, $x_i^{k^*} > 0$.

Conversely, if $x_i^{k^*} > 0$ with $k \in \bar{\mathcal{K}}$ being sufficiently large, then by (4.40) we have

$$x_i^k \geq x_i^{k^*} - c_1 \sqrt{\mu_k}. \quad (4.41)$$

From (4.39) we know that $i \in \mathcal{B}$ when $k \in \bar{\mathcal{K}}$ is sufficiently large. This and the fact that $\mu_k \rightarrow 0$ as $k \rightarrow \infty$ imply that we always have $x_i^{k^*} > c_1 \sqrt{\mu_k} + \sqrt[3]{\mu_k}$ when k is sufficiently large. Thus, by (4.41) we have $x_i^k > \sqrt[3]{\mu_k}$. \square

By combining the definitions $\mathcal{B}^k, \mathcal{N}^k, \mathcal{T}^k$ and $\mathcal{B}, \mathcal{N}, \mathcal{T}$ (see (3.8) and (4.31)) with Lemma 4.6 we may obtain

Theorem 4.7 *Let M be a sufficient matrix and Assumption 1.1 be satisfied. Let $(0, x^*, s^*)$ be an accumulation point of the iteration sequence generated by the nonstop algorithm and $\{z^k\}_{k \in \bar{\mathcal{K}} \subseteq \mathcal{K}}$ be a subsequence which converges to $z^* := (0, x^*, s^*)$. Let $\mathcal{B}^k, \mathcal{N}^k, \mathcal{T}^k$ and $\mathcal{B}, \mathcal{N}, \mathcal{T}$ be defined by (3.8) and (4.31), respectively. Then, for all sufficiently large $k \in \bar{\mathcal{K}}$,*

$$\mathcal{B}^k = \mathcal{B}, \quad \mathcal{N}^k = \mathcal{N}, \quad \text{and} \quad \mathcal{T}^k = \mathcal{T}. \quad (4.42)$$

In particular, there exist constants vectors $\hat{c}_{\mathcal{B}} > 0$ and $\hat{d}_{\mathcal{N}} > 0$ such that

$$x_{\mathcal{B}}^k > \hat{c}_{\mathcal{B}}, \quad s_{\mathcal{N}}^k > \hat{d}_{\mathcal{N}}, \quad x_{\mathcal{N}}^k \leq \sqrt[3]{\mu_k}, \quad s_{\mathcal{B}}^k \leq \sqrt[3]{\mu_k}, \quad x_{\mathcal{T}}^k \leq \sqrt[3]{\mu_k}, \quad \text{and} \quad s_{\mathcal{T}}^k \leq \sqrt[3]{\mu_k}.$$

hold for all sufficiently large $k \in \bar{\mathcal{K}}$.

Theorem 4.7, together with Remark 3.2(iv), implies the following result.

Theorem 4.8 *Let M be a sufficient matrix and let Assumption 1.1 be satisfied. If Algorithm 3.1 terminates at some iteration k_0 and if k_0 is sufficiently large, then the obtained solution is a maximally complementary solution of $\text{LCP}(M, q)$.*

Finally, we prove the finite termination property of Algorithm 3.1. We need the following proposition (see Proposition 4.1 in [16]).

Proposition 4.9 *Let $A \in \mathfrak{R}^{m \times n}$ and $\rho(A)$ denote the smallest singular value of the matrix A (Note that the singular values of A are the square roots of the nonzero eigenvalues of the matrix $A^T A$). Then for each $b \in \mathfrak{R}^m$ such that $Ax = b$ is feasible, there holds*

$$\min_{\{x \in \mathfrak{R}^n: Ax=b\}} \|x\|_2 \leq \frac{1}{\rho(A)} \|b\|_2.$$

We are now ready to show the finite termination property of Algorithm 3.1.

Theorem 4.10 *Let M be a sufficient matrix and Assumption 1.1 be satisfied. Then Algorithm 3.1 must terminate at Step 2, and hence a solution of $\text{LCP}(M, q)$ is obtained in a finite number of iterations.*

Proof. Suppose, on the contrary, that Algorithm 3.1 does not terminate at Step 2. Then, Algorithm 3.1 will generate a bounded iteration sequence $\{z^k\}$. Thus, we can assume without loss of generality that $(0, x^*, s^*)$ is accumulation point of the iteration sequence and $\{z^k\}_{k \in \bar{\mathcal{K}} \subseteq \mathcal{K}}$ is a subsequence which converges to $z^* := (0, x^*, s^*)$. Then, from Theorem 4.7 we have $\mathcal{B}^k = \mathcal{B}, \mathcal{N}^k = \mathcal{N}$, and $\mathcal{T}^k = \mathcal{T}$ for all sufficiently large $k \in \bar{\mathcal{K}}$. For simplicity, we assume in the following that $k \in \bar{\mathcal{K}}$ is sufficiently large. For any $(x^*, s^*) \in \mathcal{S}$, it can be seen that $(x_{\mathcal{B}}^*, s_{\mathcal{N}}^*)$ satisfies the constraint of subproblem (3.9), so the subproblem (3.9) is feasible. Let E denote the $n \times n$ identity matrix. From the constraint of subproblem (3.9) we have

$$\begin{bmatrix} -M_{\mathcal{B}\mathcal{B}} & E_{\mathcal{B}\mathcal{N}} \\ -M_{\mathcal{N}\mathcal{B}} & E_{\mathcal{N}\mathcal{N}} \\ -M_{\mathcal{T}\mathcal{B}} & E_{\mathcal{T}\mathcal{N}} \end{bmatrix} \begin{pmatrix} x_{\mathcal{B}} \\ s_{\mathcal{N}} \end{pmatrix} = \begin{pmatrix} q_{\mathcal{B}} \\ q_{\mathcal{N}} \\ q_{\mathcal{T}} \end{pmatrix} \quad (4.43)$$

Since $(x^k, s^k) \in \mathcal{F}$, it follows that

$$\begin{aligned} & \begin{bmatrix} -M_{\mathcal{B}\mathcal{B}} & E_{\mathcal{B}\mathcal{N}} \\ -M_{\mathcal{N}\mathcal{B}} & E_{\mathcal{N}\mathcal{N}} \\ -M_{\mathcal{T}\mathcal{B}} & E_{\mathcal{T}\mathcal{N}} \end{bmatrix} \begin{pmatrix} x_{\mathcal{B}} \\ s_{\mathcal{N}} \end{pmatrix} \\ &= - \begin{bmatrix} -M_{\mathcal{B}\mathcal{N}} & -M_{\mathcal{B}\mathcal{T}} & E_{\mathcal{B}\mathcal{N}} & E_{\mathcal{B}\mathcal{T}} \\ -M_{\mathcal{N}\mathcal{N}} & -M_{\mathcal{N}\mathcal{T}} & E_{\mathcal{N}\mathcal{N}} & E_{\mathcal{N}\mathcal{T}} \\ -M_{\mathcal{T}\mathcal{N}} & -M_{\mathcal{T}\mathcal{T}} & E_{\mathcal{T}\mathcal{N}} & E_{\mathcal{T}\mathcal{T}} \end{bmatrix} \begin{pmatrix} x_{\mathcal{N}}^k \\ x_{\mathcal{T}}^k \\ s_{\mathcal{B}}^k \\ s_{\mathcal{T}}^k \end{pmatrix} + \begin{pmatrix} q_{\mathcal{B}} \\ q_{\mathcal{N}} \\ q_{\mathcal{T}} \end{pmatrix}. \end{aligned} \quad (4.44)$$

Let $\bar{x}_{\mathcal{B}}^k = x_{\mathcal{B}} - x_{\mathcal{B}}^k$ and $\bar{s}_{\mathcal{N}}^k = s_{\mathcal{N}} - s_{\mathcal{N}}^k$. Then, by using (4.43) and (4.44), the subproblem (3.9) can be reformulated as

$$\begin{aligned} & \min_{\bar{x}_{\mathcal{B}}^k, \bar{s}_{\mathcal{N}}^k} \left\| \begin{pmatrix} \bar{x}_{\mathcal{B}}^k \\ \bar{s}_{\mathcal{N}}^k \end{pmatrix} \right\| \\ & \text{subject to } A \begin{pmatrix} \bar{x}_{\mathcal{B}}^k \\ \bar{s}_{\mathcal{N}}^k \end{pmatrix} = B \begin{pmatrix} x_{\mathcal{N}}^k \\ x_{\mathcal{T}}^k \\ s_{\mathcal{B}}^k \\ s_{\mathcal{T}}^k \end{pmatrix} \end{aligned} \quad (4.45)$$

where

$$A := \begin{bmatrix} -M_{\mathcal{B}\mathcal{B}} & E_{\mathcal{B}\mathcal{N}} \\ -M_{\mathcal{N}\mathcal{B}} & E_{\mathcal{N}\mathcal{N}} \\ -M_{\mathcal{T}\mathcal{B}} & E_{\mathcal{T}\mathcal{N}} \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} -M_{\mathcal{B}\mathcal{N}} & -M_{\mathcal{B}\mathcal{T}} & E_{\mathcal{B}\mathcal{B}} & E_{\mathcal{B}\mathcal{T}} \\ -M_{\mathcal{N}\mathcal{N}} & -M_{\mathcal{N}\mathcal{T}} & E_{\mathcal{N}\mathcal{B}} & E_{\mathcal{N}\mathcal{T}} \\ -M_{\mathcal{T}\mathcal{N}} & -M_{\mathcal{T}\mathcal{T}} & E_{\mathcal{T}\mathcal{B}} & E_{\mathcal{T}\mathcal{T}} \end{bmatrix}$$

Since the constrains of (4.45) are consistent, Proposition 4.9 implies that there is a solution sequence $\{(\tilde{x}_{\mathcal{B}}^k, \tilde{s}_{\mathcal{N}}^k)\}$ of (4.45) such that

$$\left\| \begin{pmatrix} \tilde{x}_{\mathcal{B}}^k \\ \tilde{s}_{\mathcal{N}}^k \end{pmatrix} \right\| \leq \frac{1}{\rho(A)} \left\| B \begin{pmatrix} x_{\mathcal{N}}^k \\ x_{\mathcal{T}}^k \\ s_{\mathcal{B}}^k \\ s_{\mathcal{T}}^k \end{pmatrix} \right\| \leq \frac{1}{\rho(A)} \|B\| \left\| \begin{pmatrix} x_{\mathcal{N}}^k \\ x_{\mathcal{T}}^k \\ s_{\mathcal{B}}^k \\ s_{\mathcal{T}}^k \end{pmatrix} \right\|.$$

which together with Theorem 4.7 implies that there exists a constant $\gamma > 0$ such that

$$\left\| \begin{pmatrix} \tilde{x}_{\mathcal{B}}^k \\ \tilde{s}_{\mathcal{N}}^k \end{pmatrix} \right\| \leq \gamma \sqrt[3]{\mu_k}. \quad (4.46)$$

Let $\tilde{x}^k := \hat{x}_{\mathcal{B}}^k - x_{\mathcal{B}}^k$ and $\tilde{s}^k := \hat{s}_{\mathcal{N}}^k - s_{\mathcal{N}}^k$. Then $(\hat{x}_{\mathcal{B}}^k, \hat{s}_{\mathcal{N}}^k)$ satisfies (4.43). Thus, if $(\hat{x}_{\mathcal{B}}^k, \hat{s}_{\mathcal{N}}^k) > 0$ then $((\hat{x}_{\mathcal{B}}^k, 0, 0), (0, \hat{s}_{\mathcal{N}}^k, 0))$ is a solution to $\text{LCP}(M, q)$. Hence we need to show that $(\hat{x}_{\mathcal{B}}^k, \hat{s}_{\mathcal{N}}^k) > 0$. From (4.46) we have

$$\left\| \begin{pmatrix} \hat{x}_{\mathcal{B}}^k - x_{\mathcal{B}}^k \\ \hat{s}_{\mathcal{N}}^k - s_{\mathcal{N}}^k \end{pmatrix} \right\| \leq \gamma \sqrt[3]{\mu_k}.$$

Furthermore, we have

$$\hat{x}_{\mathcal{B}}^k \geq x_{\mathcal{B}}^k - \gamma \sqrt[3]{\mu_k} e_{\mathcal{B}} \quad \text{and} \quad \hat{s}_{\mathcal{N}}^k \geq s_{\mathcal{N}}^k - \gamma \sqrt[3]{\mu_k} e_{\mathcal{N}}.$$

By using Theorem 4.7 we obtain that $\hat{x}_{\mathcal{B}}^{k_0} > 0$ and $\hat{s}_{\mathcal{N}}^{k_0} > 0$ for some sufficiently large k_0 . This indicates that Algorithm 3.1 will terminate at Step 2 in k_0 -th step. This contradiction shows that a solution to $\text{LCP}(M, q)$, $((\hat{x}_{\mathcal{B}}^k, 0, 0), (0, \hat{s}_{\mathcal{N}}^k, 0))$ must be obtained in a finite number of iterations. \square

Theorem 4.10 indicates that Algorithm 3.1 always terminates at Step 2 of the algorithm. This together with Theorem 4.8 implies that if Algorithm 3.1 terminates at the k -th iteration with k being sufficiently large, then the solution obtained is a maximally complementary solution of $\text{LCP}(M, q)$. To guarantee that k is sufficiently large at termination, one may set η sufficiently small in Algorithm 3.1, which may not be necessary in practice if the user's interest is only to find a solution, whether maximally complementary or not. However, theoretically it is clear that Algorithm 3.1 is capable to yield a maximally complementary solution of $\text{LCP}(M, q)$ in a finite number of iterations by taking η small enough.

5 Final Remarks

We have studied a smoothing algorithm for solving $\text{LCP}(M, q)$ with M being a sufficient matrix. The iteration sequence generated by the algorithm is shown to be bounded under an assumption that the LCP has a nonempty solution set. Moreover, under this assumption we prove that the algorithm may generate a maximally complementary solution of $\text{LCP}(M, q)$ in a finite number of iterations. The assumption used in this paper is weaker than those required by most of the current smoothing algorithms in the literature, including those algorithms which possess the finite termination property (see, for example, [9, 14, 15]).

The main feature of the algorithm is that a projection-type testing procedure is embedded into the Newton algorithm. This is the key for us to obtain the finite termination property of the algorithm. This procedure can be applied to different existing algorithms to obtain the finite termination property. For example, if we use the projection-type testing procedure to replace the predictor step in the predictor-corrector smoothing algorithms (see, [4, 12, 13]) or to replace the approximate Newton step in the “approximate step – centering step” algorithm (see, [8]), then the obtained algorithms can be shown to possess the finite termination property in a similar way to the one given in this paper.

References

- [1] Andreani, R. and Martinez, J.M., On the reformulation of nonlinear complementarity problems using the Fischer-Burmeister function, *Applied Mathematics Letters*, 12 (1999), 7-12.
- [2] Billup, S.C., Dirkse, S.P., and Ferris, M.C., A comparison of algorithms for large-scale mixed complementarity problems, *Comput. Optim. Appl.*, 7 (1997), 3–25.
- [3] Burke, J. and Xu, S., The global linear convergence of a non-interior path-following algorithm for linear complementarity problems, *Math. Oper. Res.*, 23 (1998), 719–734.
- [4] Burke, J. and Xu, S., A non-interior predictor-corrector path following algorithm for the monotone linear complementarity problem, *Math. Program.*, 87 (2000), 113–130.

- [5] Chen, B. and Chen, X., A global and local superlinear continuation-smoothing method for $P_0 + R_0$ and monotone NCP, *SIAM J. Optim.*, 9 (1999), 624–645.
- [6] Chen, B. and Harker, P.T., A non-interior-point continuation method for linear complementarity problem, *SIAM J. Matrix Anal. Appl.*, 14 (1993), 1168–1190.
- [7] Chen, C and Mangasarian, O.L., A class of smoothing functions for nonlinear and mixed complementarity problems, *Comput. Optim. Appl.*, 5 (1996), 97–138.
- [8] Chen, B. and Xiu, N., A global linear and local quadratic non-interior continuation method for nonlinear complementarity problems based on Chen- Mangasarian smoothing functions, *SIAM J. Optim.*, 9 (1999), 605–623.
- [9] Chen, X. and Ye, Y., On homotopy-smoothing methods for variational inequalities, *SIAM J. Control Optim.*, 37 (1999), 589-616.
- [10] Chen, X. and Ye, Y., On smoothing methods for the P_0 matrix linear complementarity problems, *SIAM J. Optim.*, 11 (2000), 341–363.
- [11] Cottle, R.W., Pang, J.-S., and Stone, R.E., *The Linear Complementarity Problem*, Academic Press, Boston, 1992.
- [12] Engelke, S. and Kanzow, C., Predictor-corrector smoothing methods for the solution of linear programming, *Preprint*, Department of Mathematics, University of Hamburg, 2000.
- [13] Engelke, S. and Kanzow, C., Improved smoothing-type methods for the solution of linear programming, *Numer. Math.* 90 (2002), 487–507.
- [14] Fang, S.C., Han, J., Huang. Z.H., and İlker Birbil, Ş., On the finite termination of an entropy function based non-interior continuation method for vertical linear complementarity problems, To appear in *J. Global Optim.*
- [15] Fischer, A. and Kanzow, C., On the finite termination of an iterative method for linear complementarity problems, *Math. Program.*, 74 (1996), 279-292.
- [16] Güler, O., Hoffman, A.J., and Rothblum, U.G., Approximations to solutions to systems of linear inequalities, *SIAM J. Matrix Anal. Appl.*, 16 (1995), 688–696.
- [17] Huang, Z.H., Sufficient conditions on nonemptiness and boundedness of the solution set of the P_0 function nonlinear complementarity problem, *Oper. Res. Letters*, 30 (2002), 202–210.

- [18] Huang, Z.H., Han, J., and Chen, Z., A predictor-corrector smoothing Newton algorithm, based on a new smoothing function, for solving the nonlinear complementarity problem with a P_0 function, *J. Optim. Theory Appl.*, 117 (2003), 39–68.
- [19] Huang, Z.H., Qi, L., and Sun, D., Sub-quadratic convergence of a smoothing Newton algorithm for the P_0 - and monotone LCP, *Math. Program.*, 99 (2004), 423–441.
- [20] Huang, Z.H., Zhang, L., and Han, J., A hybrid smoothing-nonsmooth Newton-type algorithm yielding an exact solution of the P_0 -LCP, *J. Comput. Math.*, 22(2004), 797–806.
- [21] Illés, T., Peng, J.M., Roos, C., and Trelaky, T., A strongly polynomial procedure yielding a maximally complementarity solution for $P_*(\kappa)$ linear complementarity problems, *SIAM J. Optim.*, 11 (2000), 320–340.
- [22] Kanzow, C., Some noninterior continuation methods for linear complementarity problems, *SIAM J. Matrix Anal. Appl.*, 17 (1996), 851–868.
- [23] Kojima, M., Megiddo, N., Noma, T., and Yoshise, A., A unified approach to interior point algorithms for linear complementarity problems, *Lecture Notes in Computer Sciences*, 538, Springer-Verlag, Berlin, Germany, 1991.
- [24] Mehrotra, S. and Ye, Y., On finding the optimal facet of linear programs, *Math. Program.*, 62 (1993), 497–515.
- [25] Pang, J.-S., Error bounds in mathematical programming, *Math. Program.*, 79 (1997), 299–332.
- [26] Qi, L., Sun, D., and Zhou, G., A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequality problems, *Math. Program.*, 87 (2000), 1–35.
- [27] Robinson, S.M., Some continuity properties of polyhedral multifunctions, *Math. Program. Stud.*, 14 (1981), 206–214.
- [28] Tseng, P., Analysis of a non-interior continuation method based on Chen-Mangasarian smoothing functions for complementarity problems, in *Reformulation-Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*, edited by Fukushima, M. and Qi, L., Kluwer Academic Publishers, Boston, 1999, pp. 381–404.

- [29] Väliäho, H., P_* -matrices are just sufficient, *Linear Algebra Appl.*, 239 (1996), 103–108.
- [30] Ye, Y., On the finite convergence of interior-point algorithms for linear programming, *Math. Program.*, 57 (1992), 325–335.
- [31] Zhao, Y.B. and Li, D., A globally and locally superlinearly convergent non-interior-point algorithm for P_0 LCPs, *SIAM J. Optim.*, 13 (2003), 1196–1221.
- [32] Zhang, J. and Xiu, N., Identification of the optimal active set in a noninterior continuation method for LCP, *J. Global Optim.*, 26 (2003), 183–198.