A PARAMETRIC APPROACH FOR A NONLINEAR DISCRETE LOCATION PROBLEM

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Abstract.

A discrete location problem is formulated for the design of a postal service network. The cost objective of this problem includes a nonlinear concave component. A parametric integer programming algorithm is developed to find an approximate solution to the problem. The algorithm reduces the problem into a sequence of p-median problems and deals with the nonlinear cost by a node-replacement scheme. Preliminary computational results are presented.

Keywords: Facility location problem, heuristic algorithm, integer programming.

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1 Introduction

This research is motivated by a postal service network design problem in a region with dense population. The region is divided into many *post sectors.* All mail items such as letters and parcels are first delivered to a *post center,* where they are sorted and then delivered to several *delivery bases.* Each delivery base serves its surrounding post sectors. Due to historical reasons, the location, size, and service range of the delivery bases are not quite reasonable. For instance, a new town may be served by an old base that is far away from it, while a small delivery base may have to deal with an excessively large number of mail items because of a rapid population growth in its service range. It is therefore necessary to re-consider the best number of the delivery bases, together with their location, size, and service range in order to provide qualified postal service.

In designing the new postal service network there are several factors need be considered.

- The locations of the post sectors are fixed. Thus only the locations of delivery bases are under consideration.
- The monthly amount of mail items delivered from a delivery base is called the *workload* of this base. The construction, land and other cost for building a delivery base depend on the designed workload. The dependence relation can be described by a nonlinear function.
- The transportation cost from the delivery bases to the post sectors is proportional to the distance as well as the amount of mail items delivered to the post sectors. The rest of the transportation cost, which includes the cost of transportation from the post center to the bases, is less variable and thus can be thought of as a constant.
- Finally, it is realistic to assume that only a fixed number of predetermined locations, currently one at each post sectors, can be used to build new delivery bases. This, on one hand, would make the mathematical model involve discrete variables, on the other hand, should simplify the determination of the transportation path, therefore make the distance from any delivery base to any post sector available.

Based on these considerations we acquired the following data for our model-building.

- The total amount of the estimated mail items delivered to each post sector, which is measured by *"beats"*, a common word in the postal service industry
- The distance between any potential delivery base and any post sector

- The building cost of a delivery base, which includes the land cost, the construction cost, the operating cost in the life span of the base, etc., in the form of a nonlinear function of the total beats to be served by the base

- The maximum size of each delivery base.

The rest of this paper is organized as follows. The problem is formulated as a nonlinear mixed integer program in Section 2; An algorithm, parameterized in terms of the number of delivery bases, is developed in Section 3; and our preliminary computational result is presented in Section 4.

2 Formulation of the Problem

The cost objective in the original problem consists of two components. The first one is the total building cost for the delivery bases. The second one is the total transportation cost.

The first component can be further divided into two parts. One is a fixed initial cost, which must be incurred as long as a delivery base is established, no matter where it is set up and how many beats it will serve. The other is dependent on the value of the total beats served by the delivery base. This part of cost will increase with the value of beats, but the rate of increase will decrease gradually. Indeed we find that it is an increasing nonlinear concave function. In addition, in analyzing the data of the problem we also find that, compared with transportation cost, the fixed initial construction cost is rather high. As we shall see, this trait reduces the importance of the transportation cost in our models and has an impact in the design of the corresponding effective algorithm. Unlike most of typical location problems in which the transportation cost is the dominant cost component, the computational results in the last section shows that in our problem the transportation cost has no such influence as to set up more facilities close to the demand as it typically does in other location problems.

Suppose that in our problem there are n post sectors therefore n possible locations for building delivery bases (Of course in a more general model these two numbers can be different, but this difference is not essential). We introduce the 0-1 decision variables x_{ij} and y_i, where $i, j = 1, 2, ..., n$; $x_{ij} = 1$ denotes that the post sector j is served by the delivery base i and $x_{ij} = 0$ denotes that there is no such service; $y_i = 1$ denotes that a base needs to be built at site i, otherwise $y_i = 0$. The values of y_i determine the number and the location of the delivery bases, while the values of x_{ij} indicate which post sector is served by which delivery base.

The objective function is the total cost for building bases and transportation of mail delivery. This objective function should be calculated on a special time interval. Here we calculate them on the monthly base. Since the total transportation cost from the post center to the delivery bases is omitted, the post center does not play a role in the objective. Suppose that the coefficient s_i is the building costs for base i on a monthly base. Let u_{ij} be the monthly delivery cost from base i to post sector j. Then the cost objective is

$$\sum_{i=1}^{n} s_i y_i + \sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij} x_{ij}. \tag{1}$$

We indicate that s_i is comprised of two components: a nonlinear function related to the total served beats of the delivery base and a fixed initial cost associated with each candidate delivery base site. Meanwhile, the coefficient u_{ij} is proportional both to the beats and to the distance. Let function $\phi(t_i)$ represent the variable part of s_i, which is a concave function defined on all nonnegative real numbers with $\phi(0) = 0$, $\phi(t_i) \geq 0$, where t_i denote the workload of the delivery base established on site i. Let c be the fixed part of s_i. Then $s_i = c + \phi(t_i)$. Let coefficient b_j denote the beats on post sector j and let coefficient d_{ij} denote the distance between base i and post sector j. Suppose that α is the unit delivery cost per month-beat-kilometer. Then $u_{ij} = \alpha b_j d_{ij}$. Note that we have $y_i\phi(t_i) = \phi(t_i)$. Thus the cost objective function (1) can be expressed as

$$\sum_{i=1}^{n} \phi(t_i) + c\sum_{j=1}^{n} y_i + \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} b_j d_{ij} x_{ij}, \tag{2}$$

where $t_i = \sum_{j=1}^{n} b_j x_{ij},\ i = 1, ..., n$ and the concave function $\phi(t)$ satisfies

$$\phi(0) = 0 \text{ and } \phi(t) \geq 0 \text{ for } t > 0. \tag{3}$$

It can be shown that such a concave function must be non-decreasing on $t \geq 0$. Since every post sector should be linked with one and only one delivery base, and the base-built-in post sector should be served just by the delivery base which is built in it, some constraints are introduced. They are

$$\begin{cases} \sum_{i=1}^{n} x_{ij} = 1 & j = 1, ..., n \\ x_{ij} \leq y_i & i, j = 1, ..., n \\ x_{ii} = y_i & i = 1, ..., n \\ \text{with } x_{ij}, y_i = 0 \text{ or } 1 & i, j = 1, ..., n. \end{cases} \tag{4}$$

Now more constraints are considered. First there is a capacity limit on the beats value in every delivery base. Then we have

$$t_i = \sum_{j=1}^{n} b_j x_{ij} \leq \beta y_i \qquad i = 1, ..., n,$$

where β is the capacity limit for delivery bases. Further as far as the travel distance limit is concerned, an additional set of constraints

$$d_{ij} x_{ij} \leq D \qquad i, j = 1, ..., n$$

is introduced, where D is the distance limit of the serving range of each delivery base. Thus, the model can be expressed as

$$\begin{cases} \min & \sum_{i=1}^n \phi(t_i) + c\sum_{j=1}^n y_i + \alpha \sum_{i=1}^n \sum_{j=1}^n b_j d_{ij} x_{ij} \\ \text{s.t.} & \sum_{i=1}^n x_{ij} = 1 \qquad j = 1, ..., n \\ & t_i = \sum_{j=1}^n b_j x_{ij} \qquad i = 1, ..., n \\ & t_i \le \beta y_i \qquad i = 1, ..., n \\ & d_{ij} x_{ij} \le D \qquad i, j = 1, ..., n \\ & x_{ii} = y_i \qquad i = 1, ..., n \\ \text{with} & x_{ij}, y_i = 0 \text{ or } 1 \qquad i, j = 1, ..., n. \end{cases} \tag{5}$$

We call (5) the *original postal network design problem.* The problem is quite difficult to solve. One of its special cases, the so-called uncapacitated facility location problem, is NP-hard [1]. Another complication is how to handle the nonlinear term in the objective function. In the next section we proceed to develop a heuristic algorithm for it.

3 Development of the Algorithm

The constraints of the travel distance limit can be easily removed without loss of generality by re-defining the distances d_{ij} as follows

$$d_{ij} = \begin{cases} d_{ij} & \text{if } d_{ij} \le D, \\ +\infty & \text{otherwise,} \end{cases}$$

where $+\infty$ can be replaced by a large number in practical computation. In addition, suppose that the total number of delivery bases is temporarily fixed at p. Then we have the following problem.

$$\begin{cases} \min & \sum_{i=1}^n \phi(t_i) + cp + \alpha \sum_{i=1}^n \sum_{j=1}^n b_j d_{ij} x_{ij} \\ \text{s.t.} & \sum_{i=1}^n x_{ij} = 1 \qquad j = 1, ..., n \\ & t_i = \sum_{j=1}^n b_j x_{ij} \qquad i = 1, ..., n \\ & t_i \le \beta y_i \qquad i = 1, ..., n \\ & \sum_{i=1}^n y_i = p \\ & x_{ii} = y_i \qquad i = 1, ..., n \\ \text{with} & x_{ij}, y_i = 0 \text{ or } 1 \qquad i, j = 1, ..., n. \end{cases} \tag{6}$$

We intend to solve problem (5) by solving a sequence of the parameterized problems (6). That is, we shall select p such that the corresponding problem (6) produces the minimum optimal value. Note that (6) can be regarded as a nonlinear extension to the following capacitated

p-median problem

$$\begin{cases} \min & \alpha \sum_{i=1}^n \sum_{j=1}^n b_j d_{ij} x_{ij} + cp \\ \text{s.t.} & \sum_{i=1}^n x_{ij} = 1 \qquad j = 1, ..., n \\ & t_i = \sum_{j=1}^n b_j x_{ij} \qquad i = 1, ..., n \\ & t_i \leq \beta y_i \qquad i = 1, ..., n \\ & \sum_{i=1}^n y_i = p \\ & x_{ii} = y_i \qquad i = 1, ..., n \\ \text{with} & x_{ij}, y_i = 0 \text{ or } 1 \qquad i, j = 1, ..., n \end{cases} \tag{7}$$

with an additional concave cost term in the objective function. Since there are some computer packages for solving (7), it is reasonable to separate the entire solution procedure into two stages. The first stage is to omit the nonlinear factor and to solve problem (7) for a certain range of p. The second stage is to evaluate and to adjust the solutions obtained in stage 1 according to the nonlinear factor to obtain an overall approximate optimal solution of (6). Although the motivation to stage 1 is only for solving problem (7), some of considerations for interaction from the nonlinear factor are incorporated. In particular, stage 1 will provide an estimative interval $[\underline{m}, \bar{m}]$ for the optimal value of p. We shall demonstrate that later in stage 2, when the nonlinear concave factor is added to the model, we still only need to solve problems (6) in this interval to get an approximate optimal solution for the destination problem (5).

Stage 2 of the algorithm is designed to get an approximate optimal solution for the capacitated p-median problem with a nonlinear concave term in its objective function, Namely the nonlinear term $\sum \phi(t_i)$ is added to the objective of (7). This algorithm is an improvement algorithm. Its basic process is to try to improve the optimal solution of the capacitated p-median problem according to the nonlinear concave factor in the problem. The essence of this algorithm comes from the node partition approach of Maranzana [3] and the node substitution approach of Teitz and Bart [4] for basic uncapacitated p-median problems. However, in our design we set up our new criteria for node partition and substitution according to the properties of the nonlinear concave function in our problem. With this heuristic algorithm, we approximately solve problems (6) in interval $[\underline{m}, \bar{m}]$, where the interval $[\underline{m}, \bar{m}]$ is produced by stage 1.

In dealing with problem (7) we often refer to its linear relaxation namely the corresponding linear program by replacing the integer variables with nonnegative continuous variables. That is the problem

$$\begin{cases} \min & \alpha \sum_{i=1}^n \sum_{j=1}^n b_j d_{ij} x_{ij} + cp \\ \text{s.t.} & \sum_{i=1}^n x_{ij} = 1 \qquad j = 1, ..., n \\ & t_i = \sum_{j=1}^n b_j x_{ij} \qquad i = 1, ..., n \\ & t_i \leq \beta y_i \qquad i = 1, ..., n \\ & \sum_{i=1}^n y_i = p \\ & x_{ii} = y_i, x_{ij} \geq 0 \qquad i, j = 1, ..., n. \end{cases} \tag{8}$$

For a fixed p let $G(p), g(p)$, and $\bar{g}(p)$ be the optimal values of problems (6), (7), and (8), respectively. It is well known that $\bar{g}(p)$ is a convex function in p and $\bar{g}(p) \leq g(p)$. The following proposition gives an upper bound for the minimizer of $g(p)$ in terms of the gap between (7) and (8).

Proposition 3.1 *Suppose that $g(p) - \bar{g}(p) < \epsilon$ for some p and suppose that $g(p-1) \leq g(p)$. Then we have*

$$g(p+k) > g(p-1) - (k+1)\epsilon \quad \forall\ k \leq n-p. \tag{9}$$

Proof. Since $\bar{g}(p)$ is convex, we have

$$\bar{g}(p) \leq \frac{1}{k+1}\bar{g}(p+k) + \frac{k}{k+1}\bar{g}(p-1). \tag{10}$$

We also have

$$\bar{g}(p+k) \leq g(p+k) \quad \text{and} \quad \bar{g}(p-1) \leq g(p-1). \tag{11}$$

The first assumption is

$$g(p) - \bar{g}(p) < \epsilon. \tag{12}$$

Combining (10), (11) with (12), we have

$$g(p) < \frac{1}{k+1}g(p+k) + \frac{k}{k+1}g(p-1) + \epsilon. \tag{13}$$

Now using the relationship $g(p-1) \leq g(p)$ in (13), we obtain (9). □

Proposition 3.1 provides a way of finding an upper bound $\bar{m}$ for possible optimal value of p in problem (7). That is, once we find that $g(p_0 - 1) < g(p_0)$ and $\epsilon = 0$, then the p minimizing $g(p)$ must be less than p_0. So we define

$$\bar{m} := p_0 - 1. \tag{14}$$

In the case of ϵ being small, the proposition also suggests that $\bar{m}$ be used as a heuristic bound. There is a natural lower bound for the minimizer of $g(p)$,

$$\underline{m} := \lceil B/\beta \rceil.\ \text{i.e. the minimum integer not less than } B/\beta \tag{15}$$

These bounds greatly reduce the time for stage 1 of the algorithm. Of course, if ϵ is not small, we could always take $\bar{m} = n$ as the natural upper bound.

We next provide an estimate for $G(p)$ in terms of $g(p)$.

Proposition 3.2 *Let $B = \sum_{i=1}^{n} b_i$ be the total number of beats in the system. Let $\tilde{t}_i,\ i = 1, ..., n$ be optimal to (6). Then one has*

$$\sum_{\tilde{t}_i > 0} \phi(\tilde{t}_i) \leq G(p) - g(p) \leq p\phi(B/p). \tag{16}$$

Proof. Let $\tilde{x}_{ij},\ i,j=1,...,n$ be optimal to (6). Then

$$G(p)=\sum_{i=1}^{n}\phi(\tilde{t}_i)+cp+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b_jd_{ij}\tilde{x}_{ij}\geq\sum_{\tilde{t}_i>0}\phi(\tilde{t}_i)+g(p).$$

Here we used the fact that the feasible sets of problems (6) and (7) are identical.

Let $\bar{x}_{ij},\bar{t}_i\ i,j=1,...,n$ be optimal to (7). Note that $\sum_{i=1}^{n}\bar{t}_i=B$ and only p out of the n terms are nonzero. Without loss of generality we may assume $\sum_{i=1}^{p}\bar{t}_i=B$. By concavity of ϕ we have

$$p\phi(B/p)+g(p)\geq\sum_{i=1}^{p}\phi(\bar{t}_i)+cp+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b_jd_{ij}\bar{x}_{ij}\geq G(p).$$

The result (16) follows. □

Proposition 3.2 says that $G(p)$ basically follows the trend of $g(p)+p\phi(B/p)$ if $\sum_{\tilde{t}_i>0}\phi(\tilde{t}_i)\approx p\phi(B/p)$. This is true if most of the nonzero $\tilde{t}_i$s are nearly identical, which is the case of the postal location problem according to Proposition 3.4 below. This estimate can be used to improve the lower bound $\underline{m}$ used in stage 2. That is

$$\underline{m}:=\max\{\lceil B/\beta\rceil,\text{argmin}\,[g(p)+p\phi(B/p)]\},\tag{17}$$

where "argmin" designates "the minimizer of".

This estimate also indicates that the number $\bar{m}$, originally created to upper bound the minimizer of $g(p)$, can be used to upper bound the minimizer of $G(p)$, too. See the following result.

Proposition 3.3 *If p_1 and p_2 are positive integers satisfying $p_1\leq p_2$ and if $\phi(t)$ is concave with $\phi(0)=0$, then*

$$p_1\phi(B/p_1)\leq p_2\phi(B/p_2).\tag{18}$$

Consequently, if $g(p)\geq g(\bar{m})$ for $p\geq\bar{m}$, then

$$g(p)+p\phi(B/p)\geq g(\bar{m})+\bar{m}\phi(B/\bar{m}).$$

Proof. Since $\phi(t)$ is concave and $\phi(0)=0$, we obtain

$$\frac{p_1}{p_2}\phi(B/p_1)=\frac{p_1}{p_2}\phi(B/p_1)+\left(1-\frac{p_1}{p_2}\right)\phi(0)\leq\phi(B/p_2).$$

The rest of the proposition readily follows. □

The following proposition explains why most of the nonzero $\tilde{t}_i$s tend to be identical if the concave cost term is dominant.

Proposition 3.4 *Given positive numbers β and B satisfying $B = k\beta + \beta_0$, where k is a positive integer and $0 \le \beta_0 < \beta$. Then we have*

$$\min\left\{\sum_{i=1}^{n} \phi(t_i) | \sum_{i=1}^{n} t_i = B, 0 \le t_i \le \beta,\ i = 1, ..., n\right\} = k\phi(\beta) + \phi(\beta_0).$$

Proof. First we note that for any concave function ϕ one has

$$\phi(b) + \phi(c) \ge \phi(a) + \phi(d) \tag{19}$$

if $a \le b \le c \le d$ and $a + d = b + c$.

Without loss of generality we assume that $t_1 \ge t_2 \ge \cdots \ge t_n$. According to the conditions on B and β it is obvious that $n \ge k$. The case of $n = k$ implies that $t_1 = t_2 = \cdots = t_n = \beta$ and $\beta_0 = 0$. It is readily seen that the proposition holds for this case. Thus in the following we assume that $n \ge k + 1$.

To prove this proposition we make induction on n.

When $n = 2$, since $n \ge k + 1$, we must have $k = 1$. applying (19) with

$$\beta_0 \le t_2 \le t_1 \le \beta,$$

we have

$$\phi(t_1) + \phi(t_2) \ge \phi(\beta) + \phi(\beta_0).$$

Thus, the proposition is true for $n = 2$. Suppose that the proposition is true for $n - 1$. We next prove it for $n(n > 2)$.

We consider two cases. One is $t_1 > \beta_0$, the other is $t_1 \le \beta_0$.

In the case of $t_1 > \beta_0$ we have

$$\beta_0 \le \beta_0 + (\beta - t_1) < \beta \tag{20}$$

and

$$\beta_0 \le \min\{\beta_0 + \beta - t_1, t_1\} \le \max\{\beta_0 + \beta - t_1, t_1\} \le \beta.$$

Applying (19), we have

$$\phi(\beta_0 + \beta - t_1) + \phi(t_1) \ge \phi(\beta) + \phi(\beta_0). \tag{21}$$

Now we observe that $t_2 + t_3 + \cdots + t_n = (k-1)\beta + (\beta + \beta_0 - t_1)$. According to (20), (21), and the inductive supposition on $n - 1$, we have

$$\sum_{i=1}^{n} \phi(t_i) \quad = \quad \sum_{i=2}^{n} \phi(t_i) + \phi(t_1)$$

$$
\begin{aligned}
&\geq (k-1)\phi(\beta) + \phi(\beta_0 + \beta - t_1) + \phi(t_1) \\
&\geq (k-1)\phi(\beta) + \phi(\beta_0) + \phi(\beta) \\
&= k\phi(\beta) + \phi(\beta_0).
\end{aligned}
$$

Now consider the case of $t_1 \leq \beta_0$. We have

$$0 \leq \beta_0 - t_1 < \beta \tag{22}$$

and

$$0 \leq \min\{\beta_0 - t_1, t_1\} \leq \max\{\beta_0 - t_1, t_1\} \leq \beta_0.$$

Since $\phi(0) = 0$, by (19) we have

$$\phi(\beta_0 - t_1) + \phi(t_1) \geq \phi(0) + \phi(\beta_0) = \phi(\beta_0). \tag{23}$$

Now we observe that $t_2 + \cdots + t_n = k\beta + (\beta_0 - t_1)$. According to (22), (23), and the inductive supposition on $n-1$, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \phi(t_i) &= \sum_{i=2}^{n} \phi(t_i) + \phi(t_1) \\
&\geq k\phi(\beta) + \phi(\beta_0 - t_1) + \phi(t_1) \\
&\geq k\phi(\beta) + \phi(\beta_0).
\end{aligned}
$$

Thus in both cases the proposition is true for n. The proof is completed. □

This proposition implies that if only the nonlinear function $\sum \phi(t_i)$ is concerned in the objective, then the best way is to select $k+1$ delivery bases with k of them having workload equal to the capacity limit. Now in stage 2 we can expect to improve the objective value by arranging the workload of the delivery bases, namely moving a node (from now on we call a post sector a node for simplicity) from one delivery base to another, making the latter have its workload closer to the capacity limit.

The main approach for this improvement is to find one node to shift its allocation relationship from the old delivery base to a new delivery base, while reducing the total cost. In addition, after this replacement the workload in the new delivery base should not exceed the capacity limit. This replacement will be done repeatedly until the total cost can not be further reduced. During every step of replacement, it is necessary to consider whether the sites of some delivery bases should be removed or relocated. Thus basically the entire improvement procedure is an alternating procedure between the replacement of node allocation relationships and the change of delivery bases.

We define a concept called *the neighborhood of a delivery base* in any feasible solution of a location problem. This term denotes a set of nodes which includes a node for the delivery base and all other nodes served by this facility. Therefore for any feasible solution of problem (6) all n nodes can be divided into p separate neighborhoods. For convenience we define the workload in any delivery base as *the neighborhood's demand* of the corresponding neighborhood (the synonym for the sum of the total beats of all nodes in this neighborhood).

It is obvious that in the optimal solution each neighborhood constitutes an independent 1-median sub-problem. Surely, there are many software packages for it. Therefore, whenever one neighborhood is changed, we shall use this exterior procedure to relocate the delivery base in this neighborhood at once.

In the following discussion we introduce two simplified terms. The *in-neighborhood* denotes the neighborhood which is currently under the consideration to add one node into it. The *out-neighborhood* denotes the neighborhood which is currently under the consideration to remove one node from it. In fact, in a replacement iteration we just select one in-neighborhood and one out-neighborhood.

We can see that after one iteration of node replacement, only the in-neighborhood and the out-neighborhood are changed. Other $p-2$ neighborhoods are not changed. For the neighborhood which is not changed, the corresponding 1-median problem in this neighborhood remains the same. It means that we need not consider the change of delivery base in them, and need only to solve the corresponding 1-median problems for the new in-neighborhood and for the new out-neighborhood. Every iteration in this improvement procedure can be completely described as follows.

The procedure in stage 2

Step 1 Find an in-neighborhood and out-neighborhood pair. Identify one node to be moved between them and go to Step 2. If no such pair can be found, stop. (The details of this step will be described later.)

Step 2 Relocate the delivery bases in these two neighborhoods after the node replacement by solving two 1-median problems. Go to Step 3.

Step 3 Check the total cost objective function $G(p)$. If the total cost is reduced, accept this replacement and the corresponding change of delivery bases and go to Step 1 to start a new iteration. Otherwise give up this replacement and go to Step 1 to find the next candidate node to move.

It should be noted that for the out-neighborhood if there is only one node in it, after this node is successfully added into another neighborhood, this out-neighborhood will disappear. Then the total number of neighborhoods will be reduced to $p-1$. After this step all neighborhoods and the corresponding facilities constitute a feasible solution for problems with $p-1$ facilities and continue this improvement procedure.

Now we discuss Step 1 in more detail. In this step we decide which two neighborhoods should be taken as the in-neighborhood and the out-neighborhood and decide which node in the out-neighborhood should be moved.

Reviewing Proposition 3.4, it is reasonable that we select a node whose beat is as large as possible, and this node should belong to the neighborhood whose demand is as small as possible. This node is then added into the neighborhood whose demand is as large as possible, as long as after this replacement the new neighborhood's demand does not exceed the capacity limit in the problem.

When we start to carry out Step 1, we divide the current solution into p neighborhoods and give a decreasing order on the neighborhood's demands. Before selecting the in-neighborhood and the out-neighborhood, check those neighborhoods whose demands have reached the capacity limit, or whose demands will exceed the capacity limit even if adding a node with the smallest beat in the remainder neighborhoods. Whenever such a neighborhood is found, lock it. Here we define a neighborhood as *locked* if in the future there is no replacement consideration for it. Otherwise, we define the neighborhood as *unlocked.* If a neighborhood is locked, never consider it again in the following steps.

In summary, the detail of Step 1 is as follows.

Step 1.1 Select the unblocked neighborhood with the maximal demand as the in-neighborhood. Select the unblocked neighborhood with the smallest demand as the out-neighborhood.

Step 1.2 In the out-neighborhood choose nodes one by one according to decreasing order of their beats. If the current replacement is given up in Step 3, choose the next node in the out-neighborhood to test the replacement. After all nodes in the out-neighborhood are tested but no replacement happens, give up this out-neighborhood and select the next neighborhood with the second smallest demand as the new out-neighborhood. Repeat the approach for the third, fourth, ... neighborhood if necessary. After the neighborhood whose demand exactly follows the demand of the current in-neighborhood is tested, but there is no replacement, shift to select a new in-neighborhood according to decreasing order of neighborhood's demands, then repeat the selection process for the out-neighborhood from the smallest neighborhood's demand again. After all neighborhoods have been tested under the current neighborhood order, but there is no any replacement, stop the entire improvement procedure.

4 Computational Experiments and Results

In this section we report the computational results for the model developed for the original postal network design problem. In our computational experiments the fixed cost for each delivery base is $c = 800$; the unit transportation cost is $\alpha = 5.2$; the capacity limit in each delivery base is $\beta = 400$; the total number of post sectors is $n = 83$; and the total beats in the system is $B = 1643$. All these data approximately simulate the magnitude of the practical problem.

The function $\phi(t)$ is a concave piecewise linear function, see its data in Table 1. The curve flattens out at tail end beyond 400.

Beats	The end cost	The slope in this interval
20	450	22.500
40	750	15.000
60	1000	12.500
80	1200	10.000
100	1375	8.750
120	1525	7.500
160	1800	6.875
200	2025	5.675
240	2200	4.375
280	2325	3.125
320	2400	1.875
400	2500	1.250

Table 1

The main procedures of our heuristic algorithms are coded with FORTRAN-77 and computations are done on an IBM PC compatible processor 486 DX. We use SITATION attached in Daskin [2] as external supporting procedure to deal with the classical standard p-median problem in the algorithm.

We find that the graph of $g(p)$ resembles a convex curve quite closely. In fact we have solved problem (7) for all 83 values of p with various capacity limits β. The slope of $g(p)$ is generally an increasing function. Out of the 83 values of p at only 6 of them violate the trait. However, at all violated points the deviations are very small. This could be explained by the well-known fact that the linear programming relaxation of the Euclidean p-median problem is very tight. The tight relaxation results in small duality gap and thus the convex tendency of $g(p)$. This, together with the large value of the fixed charge cost c, makes the curve of $g(p)$ attains its unique minimum point at $p = 11$, while in [1, 11] it is a strictly decreasing function and in [11, 83] it is a strictly increasing function.

Now we observe that what happens if the two-stage algorithm is used. We started stage 1 of the algorithm with $p = 5 = \lceil B/\beta \rceil$ and $p = 6$. The result is in Table 2.

p	$g(p)$	$g(p) + p\phi(B/p)$
5	31424	43478
6	29296	43131

Table 2

Since $g(p)$ decreases from $p = 5$ to $p = 6$, We should solve problem with $p > 6$ until $g(p-1) < g(p)$ is found. The result is in Table 3.

p	$g(p)$	$g(p) + p\phi(B/p)$
7	29293	43531
8	27856	44344
9	27511	44853
10	27219	45461
11	27169	46164
12	27181	46876

Table 3

Note that argmin $\{g(p)+p\phi(B/p)\} = 6$, so we get $\underline{m} = 6$ by (17). Since $g(p)$ become increasing from $p = 11$ to $p = 12$, we check the difference $g(12) - \bar{g}(12)$. It happens that $g(12) = \bar{g}(12)$, so we find $\bar{m} = 11$ by Proposition 3.1. Therefore we finally get $[\underline{m}, \bar{m}] = [6, 11]$ and we need only to carry on the second stage of the algorithm from $p = 6$ to $p = 11$. The results are in Table 4, where "variable cost" stands for $\sum_i \phi(t_i) + \alpha \sum_i \sum_j b_j d_{ij} x_{ij}$.

p	variable cost before stage 2	variable cost after stage 2	$G(p)$
6	37756.4	37756.3	42556
7	36782.9	36782.8	42382
8	37088.1	37039.8	43439
9	36716.0	36685.8	43885
10	36819.7	36788.0	44788
11	36652.7	36652.7	45452

Table 4

After running stage 2, it is found that at $p = 7$, $G(p)$ attains its minimal value 42382. This is the approximate optimal value for problem (5). At this time, the maximal workload in all delivery bases is 395. The computation is finished.

The computational result shows the influences of various factors in the objective as well as in the constraints. For instance, if the transportation cost were the dominant cost component in this model, then the obvious assignment of demand to delivery base would be via the nearest facility rule. This rule might have resulted in many facilities being set up to be close to demand to save on transportation cost. However, the nonlinear cost component in the objective encourages consolidation of demand into few locations to exploit the economy of scale offered by the concave cost function. If there is no other factors come into play, we would end up with a rather small number of facilities in the optimal solution. In the end, as a compromise of the two contrary tendencies, the optimal number of delivery bases is only relatively small. The results support the idea of the parametric approach. However, stage 2 appears contributing very little to the improvement of the objective function. The effectiveness of stage 1 is more obvious than that of stage 2. There are two possible explanations for this phenomenon. First, the data are collected from a small geographic area, so the transportation cost does not change much when a node moves to another neighborhood; second, the nonlinear cost is insensitive when the workloads are close to the limit; this again makes the improvement from moving around an individual node insignificant, given that the solutions provided by stage 1 already have their workload near the limit. It is our belief that stage 2 will play a more significant role in solving problems in which the transportation cost is not negligible or the nonlinear building cost is not flat near the capacity limit.

In summary, this paper presents a case of a nonlinear concave discrete location model and shows how this problem can be practically solved by using several heuristic schemes, which include parameterizing the problem in p and dealing with the concave cost function by using a node-replacement algorithm. We hope that our results can be helpful in developing effective algorithms for general discrete location problems involving concave costs.

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