A Non-Interior Continuation Algorithm for the P_0 or P_*-LCP with Strong Global and Local Convergence Properties*

Zheng-Hai Huang† and Jie Sun‡

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Abstract

We propose a non-interior continuation algorithm for the solution of the linear complementarity problem (LCP) with a P_0 matrix. The proposed algorithm differentiates itself from the current continuation algorithms by combining good global convergence properties with good local convergence properties under unified conditions. Specifically, it is shown that the proposed algorithm is globally convergent under an assumption which may be satisfied even if the solution set of the LCP is unbounded. Moreover, the algorithm is globally linearly and locally superlinearly convergent under a nonsingularity assumption. If the matrix in the LCP is a P_* matrix, then the above results can be strengthened to include global linear and local quadratic convergence under a strict complementary condition without the nonsingularity assumption.

Keywords Linear complementarity problem, non-interior continuation algorithm, global convergence, global linear convergence, local superlinear convergence.

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†Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, P.R. China. Email: zhenghaihuang@yahoo.com.cn. This author's work was also partially supported by the Scientific Research Foundation of Tianjin University for the Returned Overseas Chinese Scholars and the Scientific Research Foundation of Liu Hui Center for Applied Mathematics, Nankai University-Tianjin University.

‡Department of Decision Sciences, National University of Singapore, Singapore 119260, Republic of Singapore. Email: jsun@nus.edu.sg. Fax: (65)67792621.

1 Introduction

Given a matrix $M \in \Re^{n\times n}$ and a vector $q \in \Re^n$, the linear complementarity problem (LCP), denoted by LCP(M,q), is to find a vector $(x,s) \in \Re^{2n}$ such that

$$(x,s) \geq 0, \quad s = Mx + q, \quad \text{and} \quad x^T s = 0.$$

Throughout this paper, the solution set of LCP(M,q) is denoted by $\mathcal{S}$, i.e.,

$$\mathcal{S} := \{(x,s) \in \Re^{2n} : (x,s) \geq 0, s = Mx + q, \text{ and } x^T s = 0\},$$

where ":=" means "is defined as". We suppose that the matrix M is a P_0 or P_* matrix. Let $\mathcal{I} := \{1, 2, \ldots, n\}$. Recall that a matrix M is called a P_0 matrix if for any $x \in \Re^n$ with $x \neq 0$, there exists an index $i \in \mathcal{I}$ such that

$$x_i \neq 0 \quad \text{and} \quad x_i(Mx)_i \geq 0;$$

and that a matrix M is called a P_* matrix if there exists a nonnegative number κ such that

$$x^T(Mx) \geq -4\kappa \sum_{i\in\mathcal{I}_+(x)} x_i(Mx)_i, \quad \forall x \in \Re^n, \tag{1.1}$$

where $\mathcal{I}_+(x) := \{i \in \mathcal{I} : x_i(Mx)_i > 0\}$. The set of all matrices satisfying (1.1) is denoted by $P_*(\kappa)$, hence $P_* = \cup_{\kappa\geq 0} P_*(\kappa)$. Obviously, $P_*(0)$ is the set of positive semidefinite matrices. It is easily seen that $P_*(0) \subset P_*(\kappa) \subset P_0$ for all $\kappa \geq 0$. More detailed discussions on P_0 and P_* matrices can be found in Cottle, Pang, and Stone [12] and Kojima, Megiddo, Noma, and Yoshise [28].

An LCP(M,q) is said to be a P_0 (respectively, P_*, monotone) LCP if the matrix M is a P_0 (respectively, P_*, positive semidefinite) matrix. In this paper, we investigate the non-interior continuation algorithm for solving the P_0 or P_* LCP. A large number of non-interior continuation algorithms for solving various optimization problems has been proposed in the last decade (see, for example, Chen and Tseng [11], Engelke and Kanzow [13], and references therein). A focal point in the development of those algorithms has been to obtain good convergence properties, which can be further categorized into three aspects: Global convergence properties, global linear convergence properties, and local superlinear convergence properties.

Global convergence is a basic requirement for an iterative algorithm. In this case, one of the main issues is how to ensure the boundedness of the iteration sequence generated by the non-interior continuation algorithms. Various sufficient conditions have been presented for this purpose. See, for example, the $P_0 + R_0$ condition in Kanzow [26], the uniform P condition in Kanzow [27] and Xu [37], Condition 1.2 in Hotta and Yoshise [18], Assumption 1.1 in Chen and Chen [4], Assumption 2 in Chen and Xiu [8], Assumption A in Burke and Xu [1], the condition used in Lemma 4 in Tseng [35], and Assumption 1.1 in Huang [20]. It is known that all of the aforementioned conditions imply that the solution set of the concerned problem is bounded (see, for example, Zhao and Li [39] and Huang [19]). Recently, Zhao and Li [39] made an improvement on relaxing this requirement. Based on the analysis on the regularized central path [38], Zhao and Li [39] proposed a non-interior continuation algorithm for solving LCP(M, q) which is globally convergent under a new condition. The outstanding feature of the new condition is that it can be satisfied by an LCP(M, q) with an unbounded solution set.

Global linear convergence has been another research topic for non-interior continuation algorithms. To obtain global linear convergence of the algorithms, various conditions have been proposed in the literature. Among them, a common condition is that the Jacobian of a certain reformulated function is uniformly nonsingular (*the uniformly nonsingular assumption* for short). It is well known that the uniformly nonsingular assumption implies that $\mathcal{S}$ consists of a single element when M is a P_0 matrix. Some sufficient conditions of the uniformly nonsingular assumption can be found in Burke and Xu [1, Proposition 4.3], Chen and Chen [5, Section 5], Qi and Sun [31, Proposition 4.2], and Tseng [35, Corollary 2]. Recently, Huang [20] proposed a new condition which requires the iteration direction sequence obtained in the Newton equations is uniformly bounded. The new condition is a relaxation of the uniformly nonsingular assumption. The main feature of the new condition is that it does not imply the uniqueness of the solution of the P_0-LCP.

In order to obtain local superlinear convergence, most non-interior continuation algorithms need both the *nonsingularity assumption* (i.e., all generalized Jacobian of some reformulation function at each accumulation point of the iteration sequence are nonsingular) and the *strict complementarity condition* (i.e., each accumulation point of the iteration sequence is a strict complementary solution to the problem concerned). See, for example, Burke and Xu [2], Chen and Chen [5], Chen and Xiu [7], Zhao and Li [39], and Huang and Han [22]). Some efforts have been made in the literature to relax the two assumptions. Chen and Xiu [8] proposed a one-step non-interior continuation algorithm for solving the monotone LCP which is locally superlinearly convergent under the nonsingularity assumption without the strict complementarity assumption. On the other hand, Tseng [36] proposed a predictor-corrector-type non-interior continuation algorithm for solving the monotone LCP which is locally superlinearly convergent only under the strict complementarity condition. Similar improvement results have also been obtained by Engelke and Kanzow [13, 14] for linear programs and Huang [20] for the monotone LCP. Recently, Huang, Qi, and Sun [24] proposed a smoothing Newton method (which is closely related to the non-interior continuation algorithm) for solving LCP(M, q), which is either locally superlinearly convergent under the nonsingularity assumption or locally quadratically convergent under the strict complementarity condition if M is a positive semidefinite matrix.

To the best of our knowledge, there have been no such non-interior continuation algorithms in the literature that combine the newest features in all three categories above, which motivates the current work. By using a modified regularized smoothing reformulation, we propose a non-interior continuation algorithm for solving the P_0-LCP that has the following convergence properties.

- The proposed algorithm is globally convergent under an assumption which may be satisfied even if $\mathcal{S}$ is unbounded.
- The proposed algorithm is globally linearly convergent under a new assumption which is a relaxation of most existing conditions. The proposed assumption does not imply the uniqueness of the solution of the P_0-LCP.

- The proposed algorithm is globally linearly and locally superlinearly convergent under the nonsingularity assumption without the strict complementarity condition.
- If M is a P_* matrix, in addition to the above convergence properties, the proposed algorithm is globally linearly and locally quadratically convergent under the strict complementarity condition without the nonsingularity assumption.

The organization of the paper is as follows. In the next section, we discuss a modified regularized smoothing reformulation of LCP(M, q) and deduce some basic properties of it. In Section 3, we present our non-interior continuation algorithm and show that the algorithm is well defined. In Section 4, we show the global convergence of the proposed algorithm. In Section 5, we show the global linear and local superlinear convergence of the proposed algorithm for solving LCP(M, q) with M being a P_0 matrix. In Section 6, we show the global linear and local quadratic convergence of the proposed algorithm for solving P_*-LCP(M, q). Some remarks are given in Section 7.

A few words about our notation are in order. All vectors are column vectors, the superscript T denotes transpose, $\Re^n$ denotes the space of $n-$dimensional real column vectors, and $\Re^n_+$ (respectively, $\Re^n_{++}$) denotes the nonnegative (respectively, positive) orthant in $\Re^n$. We denote $\mathcal{I} = \{1, 2, \ldots, n\}$. For any vector u, we denote by u_i the ith component of u and, for any $\mathcal{K} \subset \mathcal{I}$, by $u_{\mathcal{K}}$ the vector obtained after removing from u those u_i with $i \notin \mathcal{K}$. For any vectors $u, v \in \Re^n$, we write $(u^T, v^T)^T$ as (u, v) for simplicity. For any set $\mathcal{A} \in \Re^{2n}$, we denote by $\mathrm{dist}(w, \mathcal{A})$ the Euclidean distance of the vector $w \in \Re^{2n}$ to the set $\mathcal{A}$, i.e., $\mathrm{dist}(w, \mathcal{A}) = \inf_{v \in \mathcal{A}} \|w - v\|$. We denote by $\Re^{n \times n}$ the space of $n \times n$ real matrices. For any $A \in \Re^{n \times n}$, we denote $\|A\| = \max_{u \in \Re^n, \|u\|=1} \|Mu\|$. For any continuously differentiable function $g = (g_1, g_2, \ldots, g_m)^T : \Re^m \to \Re^m$, we denote its Jacobian by $g' = (\bigtriangledown g_1, \bigtriangledown g_2, \ldots, \bigtriangledown g_m)^T$, where $\bigtriangledown g_i$ denotes the gradient of g_i for $i = 1, 2, \ldots, m$. For any $\alpha, \beta \in \Re_{++}$, we write $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) to mean α/β is uniformly bounded (respectively, tends to zero) as $\beta \to 0$. For any $(x, y) \in \Re^{2n}$ with $y - x \in \Re^n_+$, we denote by $[x, y]$ the rectangular box $[x_1, y_1] \times \cdots \times [x_n, y_n]$. Let $\mathcal{K}$ denote the set of all iteration indices, i.e. $\mathcal{K} := \{0, 1, 2, \ldots\}$. For any (μ, x, s), $(\mu_k, x^k, s^k) \in \Re_+ \times \Re^{2n}$ $(k \in \mathcal{K})$, we always use the following notation throughout this paper unless stated otherwise:

$$w := (x, s), \quad w^k := (x^k, s^k), \quad z := (\mu, w) := (\mu, x, s), \quad z^k := (\mu_k, w^k) := (\mu_k, x^k, s^k).$$

For any vector $u \in \Re^n$, we also use $\mathrm{vec}\{u_i : i \in \mathcal{I}\}$ to denote the vector u and use $\mathrm{diag}\{u_i : i \in \mathcal{I}\}$ to denote the diagonal matrix whose i-th diagonal element is u_i.

2 The Smoothing Reformulation and Its Basic Properties

At the core of the non-interior continuation algorithm is a smoothing function. Many smoothing functions have been studied in the literature; see, for example, [9, 13, 15, 16]. In this paper, we use the smoothing function $\phi : \Re^3 \to \Re$ defined by

$$\phi(\mu, a, b) = a + (b + \mu^p a) - \sqrt{[a - (b + \mu^p a)]^2 + 4\mu^2}, \tag{2.1}$$

which $p \in (0, +\infty)$. This function is called the *regularized Chen-Harker-Kanzow-Smale (CHKS) smoothing function* [6, 26, 33]. It is easy to see that the function ϕ is continuously differentiable at any $(\mu, a, b) \in \Re^3$ with $\mu > 0$.

For any $z \in \Re^{1+2n}$ and a fixed vector $d \in \Re^n$, let

$$G(z) := \begin{bmatrix} \mu \\ F(z) \end{bmatrix}, \tag{2.2}$$

where

$$F(z) := \begin{bmatrix} s - Mx - q \\ \Phi(z) - \mu d \end{bmatrix} \tag{2.3}$$

with

$$\Phi(z) := \begin{bmatrix} \phi(\mu, x_1, s_1) \\ \vdots \\ \phi(\mu, x_n, s_n) \end{bmatrix}. \tag{2.4}$$

From (2.1), it is easy to see that

$$\phi(0, a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

Thus, by (2.2), (2.3), and (2.4) we have

$$G(z) = 0 \quad \Longleftrightarrow \quad \mu = 0 \ \text{ and } \ w \text{ solves } \mathrm{LCP}(M, q). \tag{2.5}$$

We can then solve LCP(M, q) by reformulating LCP(M, q) as a system of smooth equations $G(z) = 0$ and iteratively solving this equation as $\mu \to 0$.

The function given in (2.2) is a modification of the regularized reformulation given in [38]. As we will see later, this modification is a key for us to obtain stronger local convergence without destroying the good global convergence property established in [38]. In the following we give a condition for global convergence and establish some properties of the trajectory for the LCP defined by $F(z) = 0$.

For given $p \in (0, +\infty)$, $d \in \Re^n$, and $\mu \in (0, 1]$, we define a mapping $\mathcal{F}_{(p,d,\mu)} : \Re^{2n} \to \Re^{2n}$ as follows:

$$\mathcal{F}_{(p,d,\mu)}(x, s) = \begin{bmatrix} Xs \\ s - M(x + \mu d/2) - q - \mu^p x \end{bmatrix}, \tag{2.6}$$

where $X = \mathrm{diag}\{x_i : i \in \mathcal{I}\}$.

Condition 2.1 *For given $p \in (0, +\infty)$, $d \in \Re^n$, and some constant $\hat{t} \in (0, \infty)$, there exists a constant $\hat{\mu} \in (0, 1]$ such that*

$$\cup_{\mu \in (0, \hat{\mu}]} \mathcal{F}^{-1}_{(p,d,\mu)}(D_\mu)$$

is bounded, where

$$\mathcal{F}^{-1}_{(p,d,\mu)}(D_\mu) := \{(x, s) \in \Re^{2n}_{++} : \mathcal{F}_{(p,d,\mu)}(x, s) \in D_\mu\}$$

and $D_\mu := [0, \mu e] \times [-\mu \hat{t} e, \mu \hat{t} e] \subseteq \Re^n_+ \times \Re^n$ with e being the n-vector of ones.

In a similar way to [38, Proposition 2.1], we can prove

Proposition 2.2 *Suppose that M is a P_0 matrix. If the solution set of LCP(M, q) is nonempty and bounded, then Condition 2.1 is satisfied. .*

It is known that the assumption of the solution set of LCP(M, q) being nonempty and bounded is weaker than most conditions used in the literature [19, 39]. Proposition 2.2 says that Condition 2.1 is not stronger than the assumption of the solution set of LCP(M, q) being nonempty and bounded. In fact, if $p \in (0, 1]$, then the words *"not stronger"* can be replaced by *"strictly weaker"*. This can be seen by investigating the following example.

Example 2.1 *([10, Example 1.1]) Consider LCP(M, q) with*

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

From [10] we know that M is a P_0-matrix and the solution set of this LCP(M, q) is unbounded. However, we can show that Condition 2.1 for $p \in (0, 1]$ is satisfied (The proof can be found in [25]).

Theorem 2.3 *Let M be a P_0-matrix. Denote $\mathcal{P} := \{w \in \Re^{2n} : F(z) = 0 \text{ where } \mu \in (0, 1]\}$. Then the following results hold.*

(i) For each $\mu \in (0, 1]$, there is a unique $w(\mu) \in \mathcal{P}$.

(ii) The trajectory $\mathcal{P}$ is smooth with respect to μ.

(iii) If Condition 2.1 is satisfied, then the entire trajectory $\{w(\mu) : \mu \in (0, 1]\}$ is bounded. Hence there exists at least a subsequence $\{w(\mu_k)\}$ converging to $(0, w^*)$ as $\mu_k \to 0$, where w^* is a solution to LCP(M, q).

The proof of Theorem 2.3 is quite similar to the one in [38, Theorem 4.2]. We omit it for brevity. Theorem 2.3 provides a theoretical basis for designing a continuation algorithm by using the modified regularized reformulation.

Before describing our continuation algorithm, we give several propositions which are useful for our analysis later. The proofs of these propositions are easy and we therefore omit them.

Proposition 2.4 *Let G' denote the Jacobian of the function G defined by (2.2). Then,*

$$G'(\mu, x, s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -M & I \\ \Phi^0_\mu + \Phi_\mu - d & D^0(z) + D(z) & E(z) \end{bmatrix}, \tag{2.7}$$

where I denotes the $n \times n$ identity matrix, and

$$\begin{array}{lll}
\Phi_\mu^0 := \mathrm{vec}\{(d_\mu^0)_i : i \in \mathcal{I}\}, & \text{where } (d_\mu^0)_i = p\mu^{p-1}x_i + \frac{p\mu^{p-1}x_i[x_i-(s_i+\mu^p x_i)]}{\sqrt{[x_i-(s_i+\mu^p x_i)]^2+4\mu^2}} & i \in \mathcal{I}, \\
\Phi_\mu := \mathrm{vec}\{(d_\mu)_i : i \in \mathcal{I}\}, & \text{where } (d_\mu)_i = -\frac{4\mu}{\sqrt{[x_i-(s_i+\mu^p x_i)]^2+4\mu^2}} & i \in \mathcal{I}, \\
D^0(z) := \mathrm{diag}\{(d_x^0)_i : i \in \mathcal{I}\}, & \text{where } (d_x^0)_i = \mu^p + \frac{\mu^p[x_i-(s_i+\mu^p x_i)]}{\sqrt{[x_i-(s_i+\mu^p x_i)]^2+4\mu^2}} & i \in \mathcal{I}, \\
D(z) := \mathrm{diag}\{(d_x)_i : i \in \mathcal{I}\}, & \text{where } (d_x)_i = 1 - \frac{x_i-(s_i+\mu^p x_i)}{\sqrt{[x_i-(s_i+\mu^p x_i)]^2+4\mu^2}} & i \in \mathcal{I}, \\
E(z) := \mathrm{diag}\{(d_s)_i : i \in \mathcal{I}\}, & \text{where } (d_s)_i = 1 + \frac{x_i-(s_i+\mu^p x_i)}{\sqrt{[x_i-(s_i+\mu^p x_i)]^2+4\mu^2}} & i \in \mathcal{I}.
\end{array}$$

If M is a P_0 matrix, then G' is nonsingular at any point $z \in \Re^{1+2n}$ with $\mu > 0$.

Proposition 2.5 *For any $(\mu, a, b, c) \in \Re_+ \times \Re \times \Re \times \Re$, we have*

$$\phi(\mu, a, b) = c \iff a - \frac{1}{2}c > 0,\ b + \mu^p a - \frac{1}{2}c > 0, \text{ and } (a - \frac{1}{2}c)(b + \mu^p a - \frac{1}{2}c) = \mu^2.$$

Proposition 2.6 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Let $\breve{\mathcal{S}} \subseteq \Re_+ \times \Re_+ \times \Re \times \Re \times \Re$ be a bounded set and let $(\mu_1, \mu_2, a, b, c) \in \breve{\mathcal{S}}$. Then there exists a constant $\mathcal{C}_0 > 0$ such that $|\phi(\mu_1, a, b) - \phi(\mu_2, a, b)| \leq \mathcal{C}_0|\mu_1 - \mu_2|$.*

Proposition 2.7 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Let $\tilde{S} \subseteq \Re_{++} \times \Re^2$ be a bounded set. For any $\alpha \in (0, 1)$ and $c, \hat{c} \in \tilde{S}$, there exists a constant $\mathcal{C}_1 > 0$ such that*

$$\|\phi''(c)\| \leq \mathcal{C}_1/\sqrt{[a - (b + \mu^3 a)]^2 + 4\mu^2} \leq \mathcal{C}_1/(2\mu) \tag{2.8}$$

and

$$|\phi(c + \alpha\hat{c}) - \phi(c) - \alpha\phi'(c)\hat{c}| \leq \mathcal{C}_1\alpha^2\|\hat{c}\|^2/(2\mu). \tag{2.9}$$

For any $z := (\mu, x, s) \in \Re \times \Re^n \times \Re^n$, let

$$\Phi_0(w) := 2\min\{x, s\}, \quad \tilde{\Phi}(z) := 2\min\{x, s + \mu^p x\}, \tag{2.10}$$

and

$$\xi(z) := \min\{|x_i - (s_i + \mu^p x_i)| : i \in \mathcal{I}\}. \tag{2.11}$$

Proposition 2.8 *Suppose that the function ϕ is defined by (2.1) with $p = 3$.*

(i) For any $z = (\mu, x, s) \in \Re_+ \times \Re^n \times \Re^n$ satisfying that there exists a $\hat{\mu} > 0$ such that $|-2\mu x_i(x_i - s_i) + \mu^4 x_i^2| \leq 1$ holds for all $i \in \mathcal{I}$ and all $\mu \in [0, \hat{\mu}]$, it follows that for all $\mu \in [0, \hat{\mu}]$,

$$\|\Phi_0(w) - \Phi(z)\| \leq \sqrt{5n/2}\mu + \sqrt{n}\mu^3\|x\|. \tag{2.12}$$

(ii) If $\xi(z) \geq \varepsilon$ for some constant $\varepsilon > 0$, then

$$\|\tilde{\Phi}(z) - \Phi(z)\| \leq (2/\varepsilon)\mu^2. \tag{2.13}$$

If $\xi(z) \geq \gamma\mu^t$ for two constants $t \in (0, 1)$ and $\gamma > 0$, then

$$\|\tilde{\Phi}(z) - \Phi(z)\| \leq (2n/\gamma)\mu^{2-t}. \tag{2.14}$$

3 The algorithm

Let functions Φ and $\tilde{\Phi}$ be defined by (2.4) and (2.10), respectively. Suppose that $\tau \in (0,1)$. Define $u : \Re^{1+2n} \to \Re^n$ by

$$u(z) := \begin{cases} \tilde{\Phi}(z) - \Phi(z) & \text{if } \mu \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

and $v : \Re^{1+2n} \to \Re^n$ by

$$v(z) := \begin{cases} \tau\mu e & \text{if } \tau\mu\sqrt{n} \leq \|u(z)\| \\ u(z) & \text{otherwise,} \end{cases} \tag{3.2}$$

where e denotes the n-vector of all ones. Suppose that $t \in (0,1)$ and $\gamma > 0$ are two constants, and that $\xi : \Re^{2n} \to \Re_+$ is defined by (2.11). We now define the function $\Upsilon : \Re^{1+2n} \to \Re \times \Re^n \times \Re^n$ as follows:

$$\Upsilon(z) := \begin{cases} (0,0,v(z)) & \text{if } \xi(z) > \gamma\mu^t \\ (0,0,0) & \text{otherwise.} \end{cases} \tag{3.3}$$

Denote

$$\Psi(z) := \Phi(z) - \mu d. \tag{3.4}$$

Now, we describe our non-interior continuation algorithm in detail.

Algorithm 3.1 *(A Continuation Algorithm of Predictor-Corrector-Type)*

Step 0 (**Initialization**) *Choose* $\delta_1, \delta_2, t \in (0,1)$, $\gamma, p \in (0,\infty)$, $\mu_0 \in (0,1]$, $x^0 \in \Re^n$, *and* $\sigma, \tau \in (0,1)$ *with* $\sigma + \tau < 1$. *Set* $s^0 := Mx^0 + q$ *and* $z^0 := (\mu_0, w^0) := (\mu_0, x^0, s^0)$. *Taking a constant vector* $d \in \Re^n$. *Choose a constant* β *such that* $\beta > 1 + \sqrt{5n/2} + \|d\|$ *and* $\|G(z^0)\| \leq \beta\mu_0$. *Set* $e^0 := (1,0,\ldots,0) \in \Re^{1+2n}$ *and* $k := 0$.

Step 1 (**Termination Criterion**) *If* $\|G(z^k)\| = 0$, *stop.*

Step 2 (**The Predictor Step**) *If* $\|G(z^k)\| > 1$, *then set* $\hat{z}^k := z^k$ *and go to Step 3; otherwise,*

compute $\Delta z^k := (\Delta\mu_k, \Delta x^k, \Delta s^k) \in \Re^{1+2n}$ *from the linear system*

$$G'(z^k)\Delta z^k = -G(z^k) + \Upsilon(z^k). \tag{3.5}$$

Set

$$\bar{z}^k := (\bar{\mu}_k, \bar{w}^k) := z^k + \Delta z^k. \tag{3.6}$$

If $G(0, \bar{w}^k) = 0$, *stop; otherwise,*

if $\|G(\mu_k, \bar{w}^k)\| > \beta\mu_k$, *then set* $\hat{z}^k := z^k$ *and go to Step 3; otherwise,* *let* ζ^k *be the maximum of the values* $1, \delta_2, \delta_2^2, \cdots$ *such that*

$$\|G(\zeta_k\mu_k, \bar{w}^k\| \leq \zeta_k\beta\mu_k \quad \text{and} \quad \|G(\delta_2\zeta_k\mu_k, \bar{w}^k)\| > \delta_2\zeta_k\beta\mu_k, \tag{3.7}$$

and set

$$\hat{\mu}_k := \zeta_k\mu_k \quad \text{and} \quad \hat{w}^k := \bar{w}^k. \tag{3.8}$$

Step 3 *(***The Corrector Step***) Compute* $\Delta\hat{z}^k := (\Delta\hat{\mu}_k, \Delta\hat{x}^k, \Delta\hat{s}^k) \in \Re^{1+2n}$ *from the linear system*

$$G'(\hat{z}^k)\Delta z^k = -G(\hat{z}^k) + (\sigma + \tau)\hat{\mu}_k e^0 + \Upsilon(\hat{z}^k). \tag{3.9}$$

Let θ_k *be the maximum of the values* $1, \delta_1, \delta_1^2, \cdots$ *such that*

$$\|\Psi(\hat{z}^k + \theta_k\Delta\hat{z}^k)\| \leq [1 - (1 - (\sigma + \tau))\theta_k]\sqrt{\beta^2 - 1}\hat{\mu}_k. \tag{3.10}$$

Step 4 *(***Update***) Set* $z^{k+1} := \hat{z}^k + \theta_k\Delta\hat{z}^k$ *and* $k := k + 1$*; return to Step 1.*

Remark 3.2 *(i) Algorithm 3.1 is designed by making use of some techniques used in [2, 3, 13, 14, 24, 39]. As we will see later, Algorithm 3.1 possesses stronger convergence properties than those obtained in [2, 3, 13, 14, 24, 39].*

(ii) Algorithm 3.1 can be easily started. In fact, we can choose any $z^0 \in (0, 1] \times \Re^{2n}$ *as the starting point of our algorithm, and then set*

$$\beta := \max\left\{2 + \sqrt{5n/2} + \|d\|,\ \|G(z^0)\|/\mu_0\right\}.$$

(iii) From the first equation in (3.5) *we have* $\Delta\mu_k = -\mu_k$*, and hence* $\bar{\mu}_k = \mu_k + \Delta\mu_k = 0$*. This fact will be used in our superlinear convergence analysis later.*

The following lemma demonstrates that Algorithm 3.1 is well defined. Some basic properties are also included in the lemma.

Lemma 3.3 *Suppose that* M *is a* P_0 *matrix. Then*

(i) Algorithm 3.1 is well-defined.

(ii) $\mu_k > 0$ *for all* $k \in \mathcal{K}$.

(iii) $z^k, \hat{z}^k \in \mathcal{N}(\beta) := \{z = (\mu, x, s) \in \Re_{++} \times \Re^n \times \Re^n : \|G(z)\| \leq \beta\mu\}$ *for all* $k \in \mathcal{K}$.

(iv) $s^k = Mx^k + q$ *for all* $k \in \mathcal{K}$.

(v) $\mu_{k+1} \leq \hat{\mu}_k \leq \mu_k$ *for each* $k \in \mathcal{K}$.

Proof. We first show that all results hold at the first iteration step (i.e., $k = 0$).

Case 1. Assume that Step 2 is accepted at $k = 0$.

(a) Since $\mu_k > 0$ when $k = 0$, by Proposition 2.4 we know that the equation (3.5) is solvable.

(b) If $G(0, \bar{w}^k) = 0$, then $\bar{w}^k$ solves LCP(M, q); otherwise, by continuity, there exist $\varepsilon > 0$ and $\breve{\mu} > 0$ such that $\|G(\mu, \bar{w}^k)\| > \varepsilon$ for all $\mu \in (0, \breve{\mu}]$. Thus, if $\|G(\mu_k, \bar{w}^k)\| \leq \beta\mu_k$, then the line search (3.7) is well defined for $k = 0$.

(c) Since $\hat{\mu}_k = \zeta_k \bar{\mu}_k > 0$, it follows from Proposition 2.4 that the equation (3.9) is solvable for $k = 0$.

(d) From $s^0 = Mx^0 + q$ and (3.5) we have $\hat{s}^k = M\hat{x}^k + q$ for $k = 0$; and from $\|G(z^0)\| \leq \beta\mu_0$ and (3.7) we have $\|G(\hat{z}^k)\| \leq \beta\hat{\mu}_k$ for $k = 0$. That is, we have

$$\hat{s}^k = M\hat{x}^k + q \quad \text{and} \quad \|G(\hat{z}^k)\| \leq \beta\hat{\mu}_k \quad \text{for } k = 0. \tag{3.11}$$

For any $\alpha \in (0, 1]$, let the function Ψ be defined by (3.4) and

$$r^k(\alpha) := \Psi(\hat{z}^k + \alpha\Delta\hat{z}^k) - \Psi(\hat{z}^k) - \alpha\Psi'(\hat{z}^k)\Delta\hat{z}^k, \tag{3.12}$$

then by (3.9), (3.11), and (3.12),

$$\begin{aligned} \|\Psi(\hat{z}^k + \alpha\Delta\hat{z}^k)\| &\leq (1-\alpha)\|\Psi(\hat{z}^k)\| + \alpha\|\Upsilon(\hat{z})\| + \|r^k(\alpha)\| \\ &\leq (1-\alpha)\sqrt{\beta^2 - 1}\hat{\mu}_k + \alpha\tau\sqrt{n}\hat{\mu}_k + \|r^k(\alpha)\| \\ &\leq [1 - (1-\tau)\alpha]\sqrt{\beta^2 - 1}\hat{\mu}_k + \|r^k(\alpha)\| \end{aligned} \tag{3.13}$$

where we use definitions of the function G (see (2.2)) and the function Ψ (see (3.4)). Since $\hat{\mu}^k > 0$ for $k = 0$, it is easy to see that the function G is continuously differentiable at $\hat{z}^k$, which together with (3.12) yields $\|r^k(\alpha)\| = o(\alpha)$. Therefore, from (3.13) it follows that there exists an $\bar{\alpha} \in (0, 1]$ such that

$$\|\Psi(\hat{z}^k + \alpha\Delta\hat{z}^k)\| \leq [1 - (1 - (\sigma + \tau))\alpha]\sqrt{\beta^2 - 1}\hat{\mu}_k$$

holds for any $\alpha \in (0, \bar{\alpha}]$. This demonstrates that (3.10) is well defined.

Thus, Algorithm 3.1 is well defined at $k = 0$ if Step 2 is accepted.

Case 2. Assume that Step 2 is rejected at $k = 0$. To show that the algorithm is well defined, it suffices to show that the equation (3.9) is solvable and that the line search (3.10) is well defined. Since, in this case, $\hat{\mu}^k > 0$ for $k = 0$ by $\hat{z}^k := z^k$, the above results can be shown in a similar way as those in cases (c) and (d).

Cases 1 and **2** together say that Algorithm 3.1 is well defined when $k = 0$. We next show that the results (ii)–(v) hold at $k = 0$.

(e) If Step 2 is accepted, then $\hat{\mu}_k = \zeta_k\mu_k > 0$; otherwise, $\hat{\mu}_k = \mu_k > 0$. That is, we always have $\hat{\mu}_k > 0$. Thus, by using the first equation in (3.9) we have

$$\mu_{k+1} = \hat{\mu}^k + \theta_k\Delta\hat{\mu}^k = [1 - (1 - (\sigma + \tau))\theta_k]\hat{\mu}^k > 0. \tag{3.14}$$

Therefore, the result (ii) holds for $k = 0$.

(f) If Step 2 is accepted, then by (3.11) we have $\hat{s}^k = M\hat{x}^k + q$; otherwise, by $\hat{z}^k = z^k$, we also have $\hat{s}^k = M\hat{x}^k + q$. This together with (3.9) implies that $s^{k+1} = Mx^{k+1} + q$ for $k = 0$, namely, the result (iv) holds for $k = 0$.

(g) By (3.7) we have $\hat{z}^k \in \mathcal{N}(\beta)$. On the other hand, by (3.10) we have $\|\Psi(z^{k+1})\| \leq [1 - (1 - (\sigma + \tau))\theta_k]\sqrt{\beta^2 - 1}\hat{\mu}_k$, and hence

$$\begin{aligned} \|G(z^{k+1})\| &= \sqrt{\mu_{k+1}^2 + \|\Psi(z^{k+1})\|^2} \\ &\leq \sqrt{\{[1 - (1 - (\sigma + \tau))\theta_k]\hat{\mu}_k\}^2 + \left\{[1 - (1 - (\sigma + \tau))\theta_k]\sqrt{\beta^2 - 1}\hat{\mu}_k\right\}^2} \\ &\leq \beta\mu_{k+1}, \end{aligned}$$

where the first inequality follows from (3.14). Therefore, the result (iii) holds for $k = 0$.

(h) We have that either Step 2 is rejected or Step 2 is accepted. For the former case, we have $\hat{\mu}_k = \mu_k$, and hence it follows from (3.14) that $\mu_{k+1} \leq \mu_k$. For the latter case, similar to the above inequality we have $\mu_{k+1} \leq \hat{\mu}_k$, and hence $\mu_{k+1} \leq \hat{\mu}_k = \zeta_k\mu_k \leq \mu_k$. Consider both cases together we obtain that the result (v) holds.

By combining cases (a)-(h), the lemma is shown to be valid at $k = 0$. Now assuming that the lemma hold for $k = m$, we can show that all results of the lemma hold for $k = m + 1$ in the same way as from (a) to (h) above. □

4 Global Convergence

The following lemma proves the boundedness of the sequence generated by Algorithm 3.1.

Lemma 4.1 *Suppose that M is a P_0 matrix. Let the sequence $\{z^k\}$ be generated by Algorithm 3.1. If Condition 2.1 holds, then the sequence $\{z^k\}$ is bounded.*

Proof. Since the sequence $\{\mu_k\}$ is non-negative and monotonically decreasing by Lemma 3.3(ii)(v), it follows that the sequence $\{\mu_k\}$ is bounded. Thus, we need to show that $\{w^k\}$ is a bounded sequence.

Construct a sequence $\{d^k\}$ by

$$d^k := [\Phi(z^k) - \mu_k d]/\mu_k, \qquad \forall k \in \mathcal{K}. \tag{4.1}$$

Using Lemma 3.3(iii)(iv) and the definition of the function G (see (2.2)) we can obtain that $\{d^k\}$ is a bounded sequence and

$$\|d^k\| \leq \sqrt{\beta^2 - 1}, \qquad k \in \mathcal{K}. \tag{4.2}$$

By Lemma 3.3(iv) it is easy to see that $\{s^k\}$ is bounded if $\{x^k\}$ is bounded. Thus we only need to show that the sequence $\{x^k\}$ is bounded. In the following, we assume that the sequence $\{x^k\}$ is unbounded and derive a contradiction.

Using (4.1) and Proposition 2.5 we have

$$\vec{x}_i^k > 0, \; \vec{s}_i^k > 0, \text{ and } \vec{x}_i^k \vec{s}_i^k = \mu_k^2, \quad \forall i \in \mathcal{I}. \tag{4.3}$$

where

$$\vec{x}^k := x^k - \frac{1}{2}\mu_k(d + d^k) \quad \text{and} \quad \vec{s}^k := s^k + \mu_k^p x^k - \frac{1}{2}\mu_k(d + d^k). \tag{4.4}$$

Since M is a P_0 matrix, it follows from [17, Lemma 1] that there exist a subsequence of $\{x^k\}$ (which is denoted by $\{x^k\}$ without loss of generality) and an index $i_0 \in \mathcal{I}$ such that $x_{i_0}^k \to \infty$ and $(Mx^k + q)_{i_0}$ is bounded from below. By using Lemma 3.3(iv), the equality in (4.3), and the second equality in (4.4) we have

$$(Mx + q)_{i_0} + \mu_k^p x_{i_0}^k - \frac{1}{2}\mu_k(d + d^k)_{i_0} = -\mu_k^2 / \vec{x}_{i_0}^k,$$

which implies that $\mu_k \to 0$ since $\{d^k\}$ is bounded, $x_{i_0}^k \to \infty$, and $(Mx^k + q)_{i_0}$ is bounded from below.

In addition, it follows from (4.4) and (4.1) that

$$\begin{aligned}
&\vec{s}^k - M(\vec{x}^k + \mu_k d/2) - q - \mu_k^p \vec{x}^k \\
&\quad = s^k + \mu_k^p x^k - \frac{1}{2}\mu_k(d + d^k) - M(\vec{x}^k + \mu_k d/2) - q - \mu_k^p \vec{x}^k \\
&\quad = Mx^k + q + \mu_k^p x^k - \frac{1}{2}\mu_k(d + d^k) - M(\vec{x}^k + \mu_k d/2) - q - \mu_k^p \vec{x}^k \\
&\quad = M(x^k - \vec{x}^k + \mu_k d/2) + \mu_k^p (x^k - \vec{x}^k) - \mu_k(d + d^k)/2 \\
&\quad = M(\mu_k(d + d^k)/2 + \mu_k d/2) + \mu_k^p(\mu_k(d + d^k)/2) - \mu_k(d + d^k)/2 \\
&\quad = \mu_k \left[M((d + d^k)/2 + d/2) + \mu_k^p (d + d^k)/2 - (d + d^k)/2 \right]
\end{aligned}$$

Since $\{\mu_k\}$ and $\{d^k\}$ are bounded, there exists a constant $\hat{t} > 0$ such that

$$\vec{s}^k - M(\vec{x}^k + \mu_k d/2) - q - \mu_k^p \vec{x}^k \in \mu_k[-\hat{t}e, \hat{t}e], \quad \forall k \in \mathcal{K}.$$

Thus, by using the equality in (4.3), the above formula, and the definition of the function $\mathcal{F}$ in (2.6) we have

$$\begin{aligned}
\mathcal{F}_{(p,d,\mu_k)}(\vec{x}^k, \vec{s}^k) &= \begin{bmatrix} \vec{X}^k \vec{s}^k \\ \vec{s}^k - M(\vec{x}^k + \mu_k d/2) - q - \mu_k^p \vec{x}^k \end{bmatrix} \\
&\in \mu_k[0, e] \times \mu_k[-\hat{t}e, \hat{t}e] =: D_{\mu_k},
\end{aligned}$$

which implies that $(\vec{x}^k, \vec{s}^k) \in \mathcal{F}^{-1}_{(p,d,\mu_k)}(D_{\mu_k})$ for all $k \in \mathcal{K}$. By using Condition 2.1, there exists a $\mu_* \in (0, 1]$ such that $\cup_{\mu \in (0,\mu_*]} \mathcal{F}^{-1}_{(p,d,\mu)}(D_\mu)$ is bounded. Since $\mu_k \to 0$ as $k \to \infty$, there exists some k_0 such that $\mu_k \leq \mu_*$ for all $k \geq k_0$. Thus,

$$\{(\vec{x}^k, \vec{s}^k)\}_{k \geq k_0} \subseteq \cup_{\mu_k \leq \mu_*} \mathcal{F}^{-1}_{(p,d,\mu_k)}(D_{\mu_k}) \subseteq \cup_{\mu \in (0,\mu_*]} \mathcal{F}^{-1}_{(p,d,\mu)}(D_\mu).$$

Since the right-hand side of the above formula is bounded; whereas its left-hand side is unbounded since $\{x^k\}$ is assumed to be unbounded, a contradiction is derived. Therefore, the sequence $\{(x^k, s^k)\}$ is bounded. □

Note that in the proof of Lemma 4.1 we mainly used the construction of the function G and the fact that $\|G(z^k)\| \le \beta\mu_k$ for all $k \in \mathcal{K}$. Since we have $\|G(\hat{z}^k)\| \le \beta\hat{\mu}_k$ for all $k \in \mathcal{K}$, a similar proof leads to that the sequence $\{\hat{z}^k\}$ is bounded.

Corollary 4.2 *Suppose that M is a P_0 matrix. Let the sequence $\{\hat{z}^k\}$ be generated by Algorithm 3.1. If Condition 2.1 holds, then the sequence $\{\hat{z}^k\}$ is bounded.*

Now, we show the global convergence of Algorithm 3.1.

Theorem 4.3 *Suppose that M is a P_0 matrix. Let sequences $\{z^k\}$ and $\{\hat{z}^k\}$ be generated by Algorithm 3.1. If Condition 2.1 is satisfied, then*

(i) $\mu_k \to 0$ as $k \to \infty$; and

(ii) every accumulation point of the sequence $\{w^k\}$ is a solution to LCP(M, q).

Proof. (i) By Lemma 3.3(ii)(v) we know that the sequence $\{\mu_k\}$ is non-negative and monotonically decreasing, and hence it is convergent. Let $\lim_{k\to\infty}\mu_k = \mu_*$. We need to show that $\mu_* = 0$. Assume that $\mu_* > 0$, we will derive a contradiction. By Corollary 4.2 we know that the sequences $\{\hat{z}^k\}$ is bounded. Since $\mu_* > 0$, it is easy to show that $\|[G'(\hat{z}^k)]^{-1}\| = O(1)$ for all $k \in \mathcal{K}$. Thus, by (3.9) and Lemma 3.3(iii) we have $\|\Delta\hat{z}^k\| = O(\hat{\mu}^k)$ for all $k \in \mathcal{K}$.

For convenience, we define a function $\psi : \Re^3 \to \Re$ which satisfies

$$\psi(\mu, x_i, s_i) := \phi(\mu, x_i, s_i) - \mu d_i, \quad i \in \mathcal{I},\ \forall(\mu, x, s) \in \Re^{1+2n}.$$

Then, by (3.4) we have $\Psi_i(z) = \psi(\mu, x_i, s_i)$ for all $i \in \mathcal{I}$ and all $(\mu, x, s) \in \Re^{1+2n}$.

For any $k \in \mathcal{K}$ and any $i \in \mathcal{I}$, let $c_i^k := (\hat{\mu}_k, \hat{x}_i^k, \hat{s}_i^k)$ and $\hat{c}_i^k := (\Delta\hat{\mu}_k, \Delta\hat{x}_i^k, \Delta\hat{s}_i^k)$ then by using Taylor expansion we can obtain that, for any $k \in \mathcal{K}$, $i \in \mathcal{I}$, $\theta \in (0, 1)$, and $t_i^k \in (0, 1)$,

$$\begin{aligned}
|\Psi_i(z^k + \theta\Delta\hat{z}^k)| &= |\psi(c_i^k + \theta\hat{c}_i^k)| \\
&= \left|\psi(c_i^k) + \theta\psi'(c_i^k)\hat{c}_i^k + \frac{\theta^2}{2}(\hat{c}_i^k)^T\psi''(c_i^k + t_i^k\theta\hat{c}_i^k)\hat{c}_i^k\right| \\
&= \left|\psi(c_i^k) + \theta\psi'(c_i^k)\hat{c}_i^k + \frac{\theta^2}{2}(\hat{c}_i^k)^T\phi''(c_i^k + t_i^k\theta\hat{c}_i^k)\hat{c}_i^k\right| \\
&\le (1-\theta)|\psi(c_i^k)| + \theta|\Upsilon_i(\hat{z}^k)| + O(1)\theta^2\|\hat{c}_i^k\|^2,
\end{aligned}$$

where the inequality follows from (2.8) since both c_i^k and $\hat{c}_i^k$ are bounded for each $k \in \mathcal{K}$ and $i \in \mathcal{I}$. Thus, for any $\theta \in (0, 1)$,

$$\|\Psi(\hat{z}^k + \theta\Delta\hat{z}^k)\| = \sqrt{\sum_{i=1}^{n}[\Psi_i(\hat{z}^k + \theta\Delta\hat{z}^k)]^2}$$

$$
\begin{aligned}
&\le \sqrt{\sum_{i=1}^{n}\left[(1-\theta)|\psi(c_i^k)|+\theta|\Upsilon_i(\hat{z}^k)|+O(1)\theta^2\|\hat{c}_i^k\|^2\right]^2} \\
&\le (1-\theta)\|\Psi(\hat{z}^k)\|+\theta\|\Upsilon(\hat{z}^k)\|+O(1)\theta^2\|\Delta\hat{z}^k\|^2 \\
&\le (1-\theta)\sqrt{\beta^2-1}\hat{\mu}_k+\theta\tau\sqrt{n}\hat{\mu}_k+O(\hat{\mu}_k)\theta^2 \\
&\le (1-(1-\tau)\theta)\sqrt{\beta^2-1}\hat{\mu}_k+O(\hat{\mu}_k)\theta^2,
\end{aligned}
$$

and hence there exists a constant $c > 0$ such that

$$\|\Psi(\hat{z}^k+\theta\Delta\hat{z}^k)\| \le (1-(1-\tau)\theta)\sqrt{\beta^2-1}\hat{\mu}_k+c\hat{\mu}_k\theta^2$$

It is easy to show that

$$(1-(1-\tau)\theta)\sqrt{\beta^2-1}\hat{\mu}_k+c\hat{\mu}_k\theta^2 \le [1-(1-(\sigma+\tau))\theta]\sqrt{\beta^2-1}\hat{\mu}_k$$

whenever $\theta \le \sqrt{\beta^2-1}\sigma/c$. Thus, by using the line search rule in (3.10) we have

$$\theta_k \ge \delta_1\sqrt{\beta^2-1}\sigma/c := \theta_* > 0$$

for all $k \in \mathcal{K}$. Thus, by the first equation in (3.9) we have

$$\mu_{k+1} = [1-(1-(\sigma+\tau))\theta_k]\hat{\mu}_k \le [1-(1-(\sigma+\tau))\theta_*]\hat{\mu}_k \le [1-(1-(\sigma+\tau))\theta_*]\mu_k$$

for all $k \in \mathcal{K}$, and hence $\mu_k \to 0$ as $k \to \infty$. This contradicts with the fact that $\mu_* > 0$. Thus, the result (i) holds.

(ii) From the result (i), a simple continuity argument leads to the result (ii). □

5 Convergence Rate Analysis for the P_0-LCP

In this section, we will show the global linear and local superlinear convergence of Algorithm 3.1 for solving the P_0-LCP.

The following assumption will be used in our analysis on the global linear convergence of Algorithm 3.1.

Assumption 5.1 *There exists a constant $\mathcal{C}_2 > 0$ such that $\|\Delta\hat{z}^k\| \le \mathcal{C}_2\hat{\mu}^k$ holds for all $k \in \mathcal{K}$.*

It is known that, in the global linear convergence analysis of existing non-interior continuation algorithms, a commonly used assumption is that $\|[G'(z)]^{-1}\| = O(1)$ for all $z \in \mathcal{N}(\beta)$. From (3.9) and the fact that $\hat{z}^k \in \mathcal{N}(\beta)$ for all $k \in \mathcal{K}$ it is easy to see that such an assumption is a sufficient condition of Assumption 5.1. A similar condition to Assumption 5.1 was proposed in [20]. Later we will see that Assumption 5.1 does not imply the uniqueness of the solution to LCP(M, q) if M is a P_0-matrix.

To simplify the analysis, in the following we establish our results for the special case of $p = 3$. A careful examination of the proofs shows that the results can be similarly derived for $p \ge 3$. The problem is open for $p < 3$.

Theorem 5.2 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Let M be a P_0 matrix and sequences $\{z^k\}$ and $\{\hat{z}^k\}$ be generated by Algorithm 3.1. If Condition 2.1 and Assumption 5.1 are satisfied, then there exists a constant $\mathcal{C}^* \in (0, 1)$ such that $\mu_{k+1} \le \mathcal{C}^*\mu_k$ holds for all $k \in \mathcal{K}$.*

Proof. By Corollary 4.2 we know that the sequence $\{\hat{z}^k\}$ is bounded. This together with Assumption 5.1 and (2.9) implies that there exists a constant $\mathcal{C}_1 > 0$ such that

$$|\phi(c_i^k + \alpha\hat{c}_i^k) - \phi(c_i^k) - \alpha\phi'(c_i^k)\hat{c}_i^k| \le \mathcal{C}_1\alpha^2\|\hat{c}_i^k\|^2/(2\hat{\mu}_k)$$

holds for all $k \in \mathcal{K}$ and all $i \in \mathcal{I}$, where $c_i^k := (\hat{\mu}_k, \hat{x}_i^k, \hat{s}_i^k)$ and $\hat{c}_i^k := (\Delta\hat{\mu}_k, \Delta\hat{x}_i^k, \Delta\hat{s}_i^k)$. Let the function r^k be defined by (3.12). Then

$$\begin{aligned}
\|r^k(\alpha)\| &= \|\Psi(\hat{z}^k + \alpha\Delta\hat{z}^k) - \Psi(\hat{z}^k) - \alpha\Psi'(\hat{z}^k)\Delta\hat{z}^k\| \\
&= \|\Phi(\hat{z}^k + \alpha\Delta\hat{z}^k) - \Phi(\hat{z}^k) - \alpha\Phi'(\hat{z}^k)\Delta\hat{z}^k\| \\
&= \sqrt{\sum_{i=1}^n \left[\phi(c_i^k + \alpha\Delta\hat{c}_i^k) - \phi(c_i^k) - \alpha\phi'(c_i^k)\Delta\hat{c}_i^k\right]^2} \\
&\le \sqrt{\sum_{i=1}^n \left[\frac{\mathcal{C}_1\alpha^2}{2\hat{\mu}_k}\|\Delta\hat{c}_i^k\|^2\right]^2} \\
&= \frac{\sqrt{n}\mathcal{C}_1\alpha^2}{2\hat{\mu}_k}\|\Delta\hat{z}^k\|^2 \\
&\le (\sqrt{n}\mathcal{C}_1\mathcal{C}_2/2)\hat{\mu}_k\alpha^2. \qquad (5.1)
\end{aligned}$$

where the second equality follows from (3.4) and the last inequality from Assumption 5.1. Let $\hat{\alpha} := 2\sigma\sqrt{\beta^2 - 1}/(\sqrt{n}\mathcal{C}_1\mathcal{C}_2) > 0$, then from (5.1) it follows that $\|r^k(\alpha)\| \le \sigma\sqrt{\beta^2 - 1}\alpha\hat{\mu}_k$ holds for all $\alpha \in (0, \hat{\alpha})$. Thus, it follows from (3.13) that for all $\alpha \in (0, \hat{\alpha})$,

$$\begin{aligned}
&\|\Psi(z^{k+1})\| - [1 - (1 - (\sigma + \tau))\alpha]\sqrt{\beta^2 - 1}\hat{\mu}_k \\
&\quad \le (1 - (1 - \tau)\alpha)\sqrt{\beta^2 - 1}\hat{\mu}_k + \|r^k(\alpha)\| - [1 - (1 - (\sigma + \tau))\alpha]\sqrt{\beta^2 - 1}\hat{\mu}_k \\
&\quad = -\sigma\sqrt{\beta^2 + 1}\alpha\hat{\mu}_k + \|r^k(\alpha)\| \\
&\quad \le 0. \qquad (5.2)
\end{aligned}$$

Assume that l is the largest non-negative number such that $\delta^l \le \hat{\alpha}$. Then by using the line search rule in (3.10), we have $\theta_k \ge \delta^l$ for all $k \in \mathcal{K}$. Let $\theta_* := \delta^l$. Then from (5.2) we obtain that

$$\|\Psi(z^{k+1})\| \le [1 - (1 - (\sigma + \tau))\theta_*]\sqrt{\beta^2 - 1}\hat{\mu}^k$$

holds for all $k \in \mathcal{K}$. Thus, by the first equation in (3.9) and Lemma 3.3(v) we have that

$$\mu_{k+1} = [1 - (1 - (\sigma + \tau))\theta_k]\hat{\mu}_k \le [1 - (1 - (\sigma + \tau))\theta_*]\hat{\mu}_k \le [1 - (1 - (\sigma + \tau))\theta_*]\mu_k \quad (5.3)$$

holds for all $k \in \mathcal{K}$. Let $\mathcal{C}^* := 1 - (1 - (\sigma + \tau))\theta_*$, then $\mathcal{C}^* \in (0, 1)$. Thus, by (5.3) we complete the proof. □

In the following, we discuss the local superlinear convergence of Algorithm 3.1.

Let $z^* := (\mu_*, w^*) := (\mu_*, x^*, s^*) \in \Re_+ \times \Re^{2n}$ be an accumulation point of the iteration sequence generated by Algorithm 3.1. Then Theorem 4.3 implies that $\mu_* = 0$ and w^* is a solution to LCP(M, q). In order to discuss the local superlinear convergence of the algorithm, we need the concept of semismoothness, which was originally introduced by Mifflin [29] for functionals. Qi and Sun [32] extended the definition of semismoothness to vector valued functions. We also need the concept of the B-differentiability of the locally Lipschitz function and some related results [30].

The following two lemmas can be easily proved, we omit them here.

Lemma 5.3 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Suppose that M is a P_0-matrix and that Condition 2.1 is satisfied. Let $t \in (0, 1)$ be given as in Algorithm 3.1 and the infinite sequence $\{z^k\}$ be generated by Algorithm 3.1. Then, $\|\Upsilon(z^k)\| = O\left(\|G(z^k)\|^{2-t}\right)$ for all sufficiently large $k \in \mathcal{K}$.*

Lemma 5.4 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Suppose that M is a P_0 matrix and that Condition 2.1 is satisfied. Let z^* be an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 3.1. If all $V \in \partial_B G(z^*)$ are nonsingular, then $\|G(\bar{z}^k)\| = O(\|G(z^k)\|^{2-t})$ for all $k \in \mathcal{K}$, where $\bar{z}^k$ is defined by (3.6).*

The following theorem give the local superlinear convergence of Algorithm 3.1.

Theorem 5.5 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Suppose that M is a P_0 matrix and that Condition 2.1 is satisfied. Let z^* be an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 3.1. If all $V \in \partial_B G(z^*)$ are nonsingular, then*

$$\mu_{k+1} = O(\mu_k^{2-t}) \tag{5.4}$$

holds for all sufficiently large $k \in \mathcal{K}$.

Proof. Since $\bar{\mu}_k = 0$, by Lemma 5.4 it follows that, for all sufficiently large $k \in \mathcal{K}$,

$$\|G(0, \hat{w}^k)\| = \|G(0, \bar{w}^k)\| = \|G(\bar{z}^k)\| = O(\|G(z^k)\|^{2-t}) = O(\mu_k^{2-t}),$$

where the last equality follows from the fact that $z^k \in \mathcal{N}(\beta)$. Thus, there exists a constant $\mathcal{C} > 0$ such that $\|G(0, \hat{w}^k)\| \le \mathcal{C}\mu_k^{2-t}$ holds for all sufficiently large $k \in \mathcal{K}$. Therefore, for all sufficiently large $k \in \mathcal{K}$,

$$\begin{aligned} \|G(\hat{z}^k)\| &\le \|G(\hat{z}^k) - G(0, \hat{w}^k)\| + \|G(0, \hat{w}^k)\| \\ &= \|G(\zeta_k\mu_k, \bar{w}^k) - G(0, \bar{w}^k)\| + \|G(0, \hat{w}^k)\| \\ &\le \left\| \begin{bmatrix} \zeta_k\mu_k \\ \bar{s}^k - M\bar{x}^k - q \\ \Phi(\bar{z}) - \zeta_k\mu_k d \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{s}^k - M\bar{x}^k - q \\ \Phi_0(\bar{w}) \end{bmatrix} \right\| + \|G(0, \hat{w}^k)\| \end{aligned}$$

$$
\begin{aligned}
&\le \left\| \begin{bmatrix} \zeta_k\mu_k \\ 0 \\ \Phi(\bar{z}) - \Phi_0(\bar{w}) - \zeta_k\mu_k d \end{bmatrix} \right\| + \|G(0,\hat{w}^k)\| \\
&\le \left(1+\sqrt{5n/2}+\|d\|\right)\zeta_k\mu_k + \sqrt{n}(\zeta_k\mu_k)^3\|\bar{x}^k\| + \mathcal{C}\mu_k^{2-t},
\end{aligned}
$$

where the last inequality follows from (2.12).

Since there exists a constant $\mathcal{C}_3 > 0$ such that $\sqrt{n}(\zeta_k)^3\mu_k^{1+t}\|\bar{x}^k\| + \mathcal{C} \le \mathcal{C}_3$ for all sufficiently large $k \in \mathcal{K}$. Thus, for all sufficiently large $k \in \mathcal{K}$,

$$
\|G(\hat{z}^k)\| \le \left(1+\sqrt{5n/2}+\|d\|\right)\zeta_k\mu_k + \mathcal{C}_3\mu_k^{2-t}.
$$

Thus, it is easy to show that

$$
\left(1+\sqrt{5n/2}+\|d\|\right)\zeta_k\mu_k + \mathcal{C}_3\mu_k^{2-t} \le \zeta_k\beta\mu_k
$$

whenever

$$
\zeta_k \ge \frac{\mathcal{C}_3}{\beta - \left(1+\sqrt{5n/2}+\|d\|\right)}\mu_k^{1-t}.
$$

Hence, by using the line search rule in (3.7) we have

$$
\delta_2\zeta_k \le \frac{\mathcal{C}_3}{\beta - \left(1+\sqrt{5n/2}+\|d\|\right)}\mu_k^{1-t},
$$

and so

$$
\zeta_k \le \frac{\mathcal{C}_3}{\delta_2\left[\beta - \left(1+\sqrt{5n/2}+\|d\|\right)\right]}\mu_k^{1-t}
$$

for all sufficiently large $k \in \mathcal{K}$. Therefore, by (3.8) and Lemma 3.3(v) we obtain that, for all sufficiently large $k \in \mathcal{K}$,

$$
\mu_{k+1} \le \hat{\mu}_k = \zeta_k\mu_k \le \frac{\mathcal{C}_3}{\delta_2\left[\beta - \left(1+\sqrt{5n/2}+\|d\|\right)\right]}\mu_k^{2-t} = O(\mu_k^{2-t}).
$$

This completes the proof. □

Remark 5.6 *(i) If all $V \in \partial_B G(z^*)$ are nonsingular, then there exist a constant $c_1 > 0$ and a sufficiently large $k_0 \in \mathcal{K}$ such that $\|[G'(\hat{z}^k)]^{-1}\| \le c_1$ holds for all sufficiently large $k \in \mathcal{K}$ with $k \ge k_0$. By (3.9) we further obtain that*

$$
\begin{aligned}
\|\Delta\hat{z}^k\| &\le \|[G'(\hat{z}^k)]^{-1}\|(\|G(\hat{z}^k)\| + \|(\sigma+\tau)\hat{\mu}_k e^0\| + \|\Upsilon(\hat{z}^k)\|) \\
&\le c_1[\beta + (\sigma+\tau)\sqrt{n} + \tau\sqrt{n}]\hat{\mu}_k
\end{aligned}
$$

for all sufficiently large $k \in \mathcal{K}$ with $k \ge k_0$. On the other hand, let $c_2 := \max\{\|[G'(\hat{z}^k)]^{-1}\| : k = 0,1,2,\ldots,k_0-1\}$. Then, using (3.9) we have $\|\Delta\hat{z}^k\| \le c_2[\beta+(\sigma+\tau)\sqrt{n}+\tau\sqrt{n}]\hat{\mu}_k$ for all $k \in \{0,1,2,\ldots,k_0-1\}$. Therefore, if all $V \in \partial_B G(z^*)$ are nonsingular, then Assumption 5.1 *is satisfied, i.e.,* $\|\Delta \hat{z}^k\| \le \mathcal{C}_2 \hat{\mu}_k$ *for all* $k \in \mathcal{K}$ *with* $\mathcal{C}_2 := \max\{c_1, c_2\}[\beta + (\sigma+\tau)\sqrt{n} + \tau\sqrt{n}]$*. Therefore, if all assumed conditions given in Theorem 5.5 are satisfied, then, by combining Theorems 5.2 and 5.5, we obtain that Algorithm 3.1 is both globally linearly and locally superlinearly convergent.*

(ii) The condition that all $V \in \partial_B G(z^*)$ *are nonsingular is a refinement of the condition that all* $V \in \partial G(z^*)$ *are nonsingular (The latter one has been used in the superlinear convergence analysis of many non-interior continuation methods and smoothing Newton methods). Such a condition has been used in superlinear convergence analysis of some Newton-type algorithms for solving nonsmooth equations [30]. Some detailed remarks about such a condition can be found in [30, Page 243].*

(iii) Zhao and Li [39] proposed a non-interior continuation algorithm for LCP(M, q) *which possesses the global convergence under an assumption which is probably satisfied even if LCP*(M, q) *has an unbounded solution set. However, the local superlinear convergence of their algorithm requires both the nonsingular assumption and strict complementary condition while no the global linear convergence result was reported. Therefore, our results obtained in this section is stronger than those in [39].*

6 Convergence Rate Analysis for the P_*-LCP

In this section, we will discuss the rate of convergence of Algorithm 3.1 for the P_*-LCP. Since a P_* matrix is a special case of a P_0 matrix, all convergence results obtained in the previous section are still valid. We will derive additional convergence results for Algorithm 3.1; namely, we will prove that Algorithm 3.1 is globally linearly and locally quadratically convergent under the strict complementarity condition without the nonsingularity assumption. By the strict complementarity condition we mean that $x^* + s^* > 0$. Obviously, w^* is a maximally complementary solution to LCP(M, q) (i.e., the number of its positive components is maximal). Note that the indices of the positive components are invariant among all maximally complementary solutions of LCP(M, q) when M is a P_* matrix. Let

$$\mathcal{B} := \{i \in \mathcal{I} : x_i^* > 0\} \quad \text{and} \quad \mathcal{N} := \{i \in \mathcal{I} : s_i^* > 0\}.$$

Then $\mathcal{B} \cup \mathcal{N} = \mathcal{I}$ and $\mathcal{B} \cap \mathcal{N} = \emptyset$. Let $\mathcal{S}_0$ denote the set consisting of all strict complementary solutions, i.e., $\mathcal{S}_0 := \{w \in \mathcal{S} : x_{\mathcal{B}} > 0 \text{ and } s_{\mathcal{N}} > 0\}$.

Lemma 6.1 *Suppose that the function* ϕ *is defined by* (2.1) *with* $p = 3$*. Suppose that* M *is a* P_* *matrix and that Condition 2.1 is satisfied. Let* z^* *be an accumulation point of the iteration sequence* $\{z^k\}$ *generated by Algorithm 3.1. If* w^* *satisfies the strict complementarity condition* $x^* + s^* > 0$*, then* $\text{dist}(w, \mathcal{S}_0) = O(\|G(z^k)\|)$ *holds for all* z^k *sufficiently close to* z^*.

By Lemma 6.1 we have that there exists a constant $\hat{\mathcal{C}} > 0$ such that

$$\|w^k - w^{k*}\| \le \hat{\mathcal{C}} \|G(z^k)\| \tag{6.1}$$

for all z^k sufficiently close to z^*.

Lemma 6.2 *If all conditions in Lemma 6.1 are satisfied, then, for all z^k sufficiently close to z^*, the following two results hold.*

(i) There exists a constant $\varepsilon > 0$ such that

$$x_i^k - (s_i^k + \mu_k^3 x_i^k) \geq \varepsilon \ \ \forall i \in \mathcal{B} \quad \text{and} \quad (s_i^k + \mu_k^3 x_i^k) - x_i^k \geq \varepsilon \ \ \forall i \in \mathcal{N}.$$

(ii) $v(z^k) = u(z^k)$, where functions v and u are defined by (3.2) *and* (3.1)*, respectively.*

Lemma 6.3 *If all conditions in Lemma 6.1 are satisfied, then*

$$\|\Delta z^k\| = O(\|G(z^k)\|) \tag{6.2}$$

for all z^k sufficiently close to z^*.

Proof. By using Lemma 3.3(iv) and Lemma 6.2(ii), we can obtain from (3.5) that, for all z^k sufficiently close to z^*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -M & I \\ \Phi^0_{\mu_k} + \Phi_{\mu_k} - d & D^0(z^k) + D(z^k) & E(z^k) \end{bmatrix} \begin{bmatrix} \Delta\mu_k \\ \Delta x^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} -\mu_k \\ 0 \\ -[\Phi(z^k) - \mu_k d] + u(z^k) \end{bmatrix}. \tag{6.3}$$

From the first equality in (6.3) it follows that, for all z^k sufficiently close to z^*,

$$|\Delta\mu_k| \leq \mu_k \leq \|G(z^k)\|. \tag{6.4}$$

In the following, we investigate the upper bound of $\|\Delta x^k\|$ and $\|\Delta s^k\|$. For any $k \in \mathcal{K}$, let

$$\widehat{\Delta x}^k := x^k + \Delta x^k - x^{k*} \quad \text{and} \quad \widehat{\Delta s}^k := s^k + \Delta s^k - s^{k*}, \tag{6.5}$$

where w^{k*} is the projection of w^k onto the set $\mathcal{S}_0$. By using the definition of the function u, Lemma 3.3(iv), (6.3), and (6.5) we can obtain that, for all z^k sufficiently close to z^*,

$$\widehat{\Delta s}^k = M\widehat{\Delta x}^k \tag{6.6}$$

and

$$[D^0(z^k) + D(z^k)]\widehat{\Delta x}^k + E(z^k)\widehat{\Delta s}^k = \alpha(z^k), \tag{6.7}$$

where $\alpha(z^k) := \varpi(z^k) + \varrho(z^k)$ with

$$\varpi(z^k) := [D^0(z^k) + D(z^k)](x^k - x^{k*}) + E(z^k)(s^k - s^{k*}) - \tilde{\Phi}(z^k) \tag{6.8}$$

and

$$\varrho(z^k) := 2[\tilde{\Phi}(z^k) - \Phi(z^k)] + \Phi_{\mu_k}\mu_k + \Phi^0_{\mu_k}\mu_k. \tag{6.9}$$

Since M is a P_* matrix, we can further obtain from (6.6) and (6.7) that, for all z^k sufficiently close to z^*,

$$
\begin{aligned}
(\widehat{\Delta x}^k)^T \widehat{\Delta s}^k &= (\widehat{\Delta x}^k)^T M \widehat{\Delta x}^k \geq -4\kappa \sum_{i \in I_+(\widehat{\Delta x}^k)} \widehat{\Delta x}_i^k (M \widehat{\Delta x}^k)_i \\
&= -4\kappa \sum_{i \in I_+(\widehat{\Delta x}^k)} \widehat{\Delta x}_i^k \widehat{\Delta s}_i^k \\
&= -4\kappa \sum_{i \in I_+(\widehat{\Delta x}^k)} \widehat{\Delta x}_i^k \left\{ [E^{-1}(z^k)\alpha(z^k)]_i - [E^{-1}(z^k)(D^0(z^k) + D(z^k))\widehat{\Delta x}^k]_i \right\}.
\end{aligned}
$$

Since, for all z^k sufficiently close to z^*,

$$
\widehat{\Delta x}_i^k [E^{-1}(z^k)(D^0(z^k) + D(z^k))\widehat{\Delta x}^k)]_i = \widehat{\Delta x}_i^k E_{ii}^{-1}(z^k)(D_{ii}^0(z^k) + D_{ii}(z^k))\widehat{\Delta x}_i^k \geq 0
$$

and

$$
\begin{aligned}
\widehat{\Delta x}_i^k [E^{-1}(z^k)\alpha(z^k)]_i &= \widehat{\Delta x}_i^k E_{ii}^{-1}(z^k)\alpha_i(z^k) \leq |\widehat{\Delta x}_i^k E_{ii}^{-1}(z^k)\alpha_i(z^k)| \\
&= \left| \left[\mathrm{diag}(\widehat{\Delta x}^k) E^{-1}(z^k)\alpha(z^k) \right]_i \right| \leq \|(\widehat{\Delta x}^k)^T E^{-1}(z^k)\alpha(z^k)\| \\
&\leq \|E^{-1}(z^k)\widehat{\Delta x}^k\| \|\alpha(z^k)\|,
\end{aligned}
$$

it follows that, for all z^k sufficiently close to z^*,

$$
(\widehat{\Delta x}^k)^T \widehat{\Delta s}^k \geq -4\kappa \sum_{i \in I_+(\widehat{\Delta x}^k)} \widehat{\Delta x}_i^k [E^{-1}(z^k)\alpha(z^k)]_i \geq -4n\kappa \|E^{-1}(z^k)\widehat{\Delta x}^k\| \|\alpha(z^k)\|.
$$

Let $r_k := \min_{i \in \mathcal{I}} \left\{ [D_{ii}^0(z^k) + D_{ii}(z^k)] E_{ii}(z^k) \right\}$. Then, for all z^k sufficiently close to z^*,

$$
\begin{aligned}
r_k \|E^{-1}(z^k)\widehat{\Delta x}^k\|^2 &\leq (\widehat{\Delta x}^k)^T E^{-1}(z^k)[D^0(z^k) + D(z^k)] E(z^k) E^{-1}(z^k) \widehat{\Delta x}^k \\
&= (\widehat{\Delta x}^k)^T E^{-1}(z^k)[D^0(z^k) + D(z^k)] \widehat{\Delta x}^k \\
&= (\widehat{\Delta x}^k)^T E^{-1}(z^k)\alpha(z^k) - (\widehat{\Delta x}^k)^T \widehat{\Delta s}^k \\
&\leq \|E^{-1}(z^k)\widehat{\Delta x}^k\| \|\alpha(z^k)\| + 4n\kappa \|E^{-1}(z^k)\widehat{\Delta x}^k\| \|\alpha(z^k)\| \\
&= (1 + 4n\kappa) \|E^{-1}(z^k)\widehat{\Delta x}^k\| \|\alpha(z^k)\|,
\end{aligned}
$$

and hence,

$$
\|E^{-1}(z^k)\widehat{\Delta x}^k\| \leq (1 + 4n\kappa)\|\alpha(z^k)\|/r_k. \tag{6.10}
$$

In order to give the upper bound of $\|E^{-1}(z^k)\widehat{\Delta x}^k\|$ by $\|G(z^k)\|$, we need to investigate $\varpi(z^k)/r_k$ and $\varrho(z^k)/r_k$.

First, we give an upper bound of $\varpi(z^k)/r_k$ by $\|G(z^k)\|$.

Since $x_i^{k*} = 0$ for all $i \in \mathcal{N}$ and $s_i^{k*} = 0$ for all $i \in \mathcal{B}$, by making use of Lemma 6.2(i) and the fact that $D(z^k) + E(z^k) = 2I$ for all $k \in \mathcal{K}$, we obtain from (6.8) that for all z^k sufficiently close to z^*,

$$\begin{aligned}\varpi_{\mathcal{B}}(z^k) &= (D^0_{\mathcal{B}\mathcal{B}}(z^k) + D_{\mathcal{B}\mathcal{B}}(z^k))(x^k_{\mathcal{B}} - x^{k*}_{\mathcal{B}}) + E_{\mathcal{B}\mathcal{B}}(z^k)(s^k_{\mathcal{B}} - s^{k*}_{\mathcal{B}}) - (2s^k_{\mathcal{B}} + 2\mu_k^3 x^k_{\mathcal{B}}) \\ &= D_{\mathcal{B}\mathcal{B}}(z^k)(x^k_{\mathcal{B}} - x^{k*}_{\mathcal{B}} - s^k_{\mathcal{B}} + s^{k*}_{\mathcal{B}}) + D^0_{\mathcal{B}\mathcal{B}}(z^k)(x^k_{\mathcal{B}} - x^{k*}_{\mathcal{B}}) - 2\mu_k^3 x^k_{\mathcal{B}},\end{aligned}$$

and similarly,

$$\varpi_{\mathcal{N}}(z^k) = E_{\mathcal{N}\mathcal{N}}(z^k)(s^k_{\mathcal{N}} - s^{k*}_{\mathcal{N}} - x^k_{\mathcal{N}} + x^{k*}_{\mathcal{N}}) + D^0_{\mathcal{N}\mathcal{N}}(z^k)(x^k_{\mathcal{N}} - x^{k*}_{\mathcal{N}}).$$

Let

$$\hat{\varpi}(z^k) := \|D^0_{\mathcal{B}\mathcal{B}}(z^k)\|\|x^k_{\mathcal{B}} - x^{k*}_{\mathcal{B}}\| + 2\mu_k^3\|x^k_{\mathcal{B}}\| + \|D^0_{\mathcal{N}\mathcal{N}}(z^k)\|\|x^k_{\mathcal{N}} - x^{k*}_{\mathcal{N}}\|, \tag{6.11}$$

we further obtain that, for all z^k sufficiently close to z^*,

$$\|\varpi(z^k)\| \le \max\{\|D_{\mathcal{B}\mathcal{B}}(z^k)\|, \|E_{\mathcal{N}\mathcal{N}}(z^k)\|\}(\|x^k - x^{k*}\| + \|s^k - s^{k*}\|) + \hat{\varpi}(z^k). \tag{6.12}$$

For any $k \ge 0$, let $g_i^k := x_i^k - (s_i^k + \mu_k^3 x_i^k)$. Lemma 6.2(i) indicates that there exists a constant $\varepsilon > 0$ such that $|g_i^k| > \varepsilon$ for all z^k sufficiently close to z^*. Thus, by definitions of $D(z^k)$ and $E(z^k)$ given in Proposition 2.4 we can further obtain that there exists a constant $\mathcal{C}_4 > 0$ such that, for all z^k sufficiently close to z^*,

$$r_k = \min_{i \in \mathcal{I}} \left\{4\mu_k^2/[(g_i^k)^2 + 4\mu_k^2] + O(\mu_k^3)\right\} \ge \mathcal{C}_4\mu_k^2, \tag{6.13}$$

Thus, it follows from (6.13) that, for any $i \in \mathcal{B}$,

$$\begin{aligned}\frac{D_{ii}(z^k)}{r_k} &\le \frac{1 - g_i^k/\sqrt{(g_i^k)^2 + 4\mu_k^2}}{\mathcal{C}_4\mu_k^2} = \frac{\sqrt{(g_i^k)^2 + 4\mu_k^2} - g_i^k}{\mathcal{C}_4\mu_k^2\sqrt{(g_i^k)^2 + 4\mu_k^2}} \\ &= \frac{4}{\mathcal{C}_4\sqrt{(g_i^k)^2 + 4\mu_k^2}\left(\sqrt{(g_i^k)^2 + 4\mu_k^2} + g_i^k\right)} = O(1)\end{aligned}$$

and that, for any $i \in \mathcal{N}$,

$$\frac{E_{ii}(z^k)}{r_k} \le \frac{4}{\mathcal{C}_4\sqrt{(g_i^k)^2 + 4\mu_k^2}\left(\sqrt{(g_i^k)^2 + 4\mu_k^2} + g_i^k\right)} = O(1).$$

In addition, by using the definition of D^0, (6.1), and (6.11) we have that, for all z^k sufficiently close to z^* and all $i \in \mathcal{I}$,

$$\frac{\hat{\varpi}(z^k)}{r^k} = \frac{O(\mu_k^3)\|x^k_{\mathcal{B}} - x^{k*}_{\mathcal{B}}\| + O(\mu_k^3) + O(\mu_k^3)\|x^k_{\mathcal{N}} - x^{k*}_{\mathcal{N}}\|}{\mathcal{C}_4\mu_k^2} = O(\|G(z^k)\|).$$

Therefore, by (6.12) we further obtain that, for all z^k sufficiently close to z^*,

$$\|\varpi(z^k)\|/r_k = O(\|x^k - x^{k*}\| + \|s^k - s^{k*}\|) + O(\|G(z^k)\|) = O(\|G(z^k)\|). \tag{6.14}$$

Next, we give the upper bound of $\varrho(z^k)/r_k$ by $\|G(z^k)\|$.

For all z^k sufficiently close to z^*, by (6.9) we have

$$\begin{aligned}
\varrho_i(z^k) &= 2\left[\sqrt{(g_i^k)^2+4\mu_k^2}-\sqrt{(g_i^k)^2}\right]+\Phi_\mu\mu_k+\Phi^0_{\mu_k}\mu_k \\
&= \frac{8\mu_k^2}{\sqrt{(g_i^k)^2+4\mu_k^2}+\sqrt{(g_i^k)^2}}-\frac{4\mu_k^2}{\sqrt{(g_i^k)^2+4\mu_k^2}}+\Phi^0_{\mu_k}\mu_k \\
&= \frac{4\mu_k^2\left[\sqrt{(g_i^k)^2+4\mu_k^2}-\sqrt{(g_i^k)^2}\right]}{\sqrt{(g_i^k)^2+4\mu_k^2}\left(\sqrt{(g_i^k)^2+4\mu_k^2}+\sqrt{(g_i^k)^2}\right)}+\Phi^0_{\mu_k}\mu_k \\
&= \frac{16\mu_k^4}{\sqrt{(g_i^k)^2+4\mu_k^2}\left(\sqrt{(g_i^k)^2+4\mu_k^2}+\sqrt{(g_i^k)^2}\right)^2}+\Phi^0_{\mu_k}\mu_k \\
&= O(\mu_k^4)+O(\mu_k^3),
\end{aligned}$$

which implies

$$\frac{\|\varrho(z^k)\|}{r_k}\le\sum_{i\in\mathcal{I}}\frac{|\varrho(z^k)_i|}{\mathcal{C}_4\mu_k^2}=O(\mu_k)=O(\|G(z^k)\|). \tag{6.15}$$

Now, by combining (6.10) with (6.14) and (6.15) we can further obtain that

$$\|E^{-1}(z^k)\widehat{\Delta x}^k\|\le(1+4n\kappa)\left\{\|\varpi(z^k)\|/r_k,\|\varrho(z^k)\|/r_k\right\}=O(\|G(z^k)\|) \tag{6.16}$$

holds for all z^k sufficiently close to z^*. Since $\|E(z^k)\|\le 2$, it follows from (6.1) that for all z^k sufficiently close to z^*,

$$\|E(z^k)^{-1}\widehat{\Delta x}^k\|\ge[\|\Delta x^k\|-\|x^k-x^{k*}\|]/\|E(z^k)\|\ge\|\Delta x^k\|-\hat{\mathcal{C}}\|G(z^k)\|/2.$$

This, together with (6.16), implies that for all z^k sufficiently close to z^*,

$$\|\Delta x^k\|=O(\|G(z^k)\|). \tag{6.17}$$

Now, we estimate $\|\Delta s^k\|$. From (6.7) we have that for all z^k sufficiently close to z^*,

$$\begin{aligned}
\|\widehat{\Delta s}^k\| &\le \|[D(z^k)+D^0(z^k)]^{-1}E(z^k)^{-1}\|\|D(z^k)+D^0(z^k)\|\|\alpha(z^k)\| \\
&\quad +\|D(z^k)+D^0(z^k)\|\|E(z^k)^{-1}\widehat{\Delta x}^k\| \\
&\le 3\frac{\|\varpi(z^k)\|+\|\varrho(z^k)\|}{\min_{i\in\mathcal{I}}\{[D_{ii}(z^k)+D^0_{ii}(z^k)]E_{ii}(z^k)\}}+3\|E(z^k)^{-1}\widehat{\Delta x}^k\| \\
&= O(\|G(z^k)\|),
\end{aligned} \tag{6.18}$$

where the second inequality is due to $\|D(z^k)+D^0(z^k)\|\le 2+\mu_k^3\|x^k\|$. Hence, it follows from (6.18) that for all z^k sufficiently close to z^*,

$$\|\Delta s^k\|\le\|s^k+\Delta s^k-s^{k*}\|+\|s^k-s^{k*}\|=O(\|G(z^k)\|). \tag{6.19}$$

Finally, by combining (6.4) with (6.17) and (6.19), we obtain that (6.2) holds for all z^k sufficiently close to z^*. $\square$

Let $\{\hat{z}^k\}$ be generated by the predictor step and let $\{\Delta\hat{z}^k\}$ be generated by the corrector step. Then from the construction of Algorithm 3.1, it is easy to see that Lemmas 6.1, 6.2, and 6.3 are still satisfied if we replace $\{z^k\}$ and $\{\Delta z^k\}$ by $\{\hat{z}^k\}$ and $\{\Delta\hat{z}^k\}$, respectively.

Corollary 6.1 *If all conditions in Lemma 6.1 are satisfied with $\{z^k\}$ and $\{\Delta z^k\}$ being replaced by $\{\hat{z}^k\}$ and $\{\Delta\hat{z}^k\}$, respectively. Then $\|\Delta\hat{z}^k\| = O(\|G(\hat{z}^k)\|)$ for all z^k sufficiently close to z^*.*

Lemma 6.4 *Suppose that all the conditions assumed in Lemma 6.1 are satisfied. Then $\|G(\bar{z}^k)\| = O(\|G(z^k)\|^2)$ holds for all z^k sufficiently close to z^*, where $\bar{z}^k = z^k + \Delta z^k$.*

Proof. Since w^* satisfies the strict complementary condition, it is easy to see from (2.8) and Lemma 6.2(i) that $\|\phi''(\mu_k, x_i^k, s_i^k)\| = O(1)$ holds for all $k \in \mathcal{K}$ and all $i \in \mathcal{I}$. Thus, by using (2.9) we have

$$\|\Phi(z^k + \Delta z^k) - \Phi(z^k) - \Phi'(z^k)\Delta z^k\| = O(\|\Delta z^k\|^2), \quad \forall k \in \mathcal{K}.$$

By using the definition of the function G we further obtain

$$\|G(z^k + \Delta z^k) - G(z^k) - G'(z^k)\Delta z^k\| = O(\|\Delta z^k\|^2), \quad \forall k \in \mathcal{K}.$$

Thus, it follows from Lemma 6.3 and the proof in Lemma 6.2(ii) that

$$\begin{aligned}
\|G(z^k + \Delta z^k)\| &= \|G(z^k + \Delta z^k) - G(z^k) - G'(z^k)\Delta z^k + G(z^k) + G'(z^k)\Delta z^k\| \\
&\le \|G(z^k + \Delta z^k) - G(z^k) - G'(z^k)\Delta z^k\| + \|G(z^k) + G'(z^k)\Delta z^k\| \\
&= O(\|\Delta z^k\|^2) + \|u(z^k)\|^2 \\
&= O(\|\Delta z^k\|^2) + O(\mu_k^2) \\
&= O(\|G(z^k)\|^2)
\end{aligned}$$

for all z^k sufficiently close to z^*. This completes the proof. $\square$

Theorem 6.5 *Suppose that the function ϕ is defined by (2.1) with $p = 3$. Suppose that M is a P_* matrix and that Condition 2.1 is satisfied. Let z^* be an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 3.1. If w^* satisfies the strict complementarity condition $x^* + s^* > 0$, then*

$$\mu_{k+1} = O\left(\mu_k^2\right)$$

holds for all sufficiently large $k \in \mathcal{K}$.

Proof. By using the results of Corollary 6.1 and Lemma 6.4, we can show the result of the theorem in a similar way to those in Theorem 5.5 and in [24, Theorem 4]. We omit it here. $\square$

Remark 6.6 *(i) If the strict complementarity condition holds, then it follows from Corollary 6.1 that there exist a constant* $c_3 > 0$ *and a sufficiently large* $k_0 \in \mathcal{K}$ *such that* $\|\Delta \hat{z}^k\| \leq c_3 \|G(\hat{z}^k)\| \leq c_3 \beta \hat{\mu}_k$ *holds for all* $k \in \mathcal{K}$ *with* $k \geq k_0$*. Thus, by using a similar discussion to the one in Remark 5.6(i) we conclude that Assumption 5.1 is satisfied, Therefore, by combining Theorems 5.2 and 6.5, we imply that Algorithm 3.1 is globally linearly and locally quadratically convergent.*

(ii) Consider Remark 5.6(i) and Remark 6.6(i) together, it is easy to see that Assumption 5.1 may be satisfied even if the solution of the P_0*-LCP is not unique.*

(iii) It is known that non-interior continuation algorithms proposed in [13, 14, 20, 36] possess the local superlinear rate of convergence under the strict complementary condition. However, the problems discussed in [13, 14, 20, 36] are monotone LCPs or linear programs. Thus, the convergence results established in this section are broader in the sense that they apply to P_**-LCP and stronger in the sense that both global linear and local quadratic, rather than only local quadratic rate of convergence, are obtained.*

7 Some Final Remarks

It is known that the non-interior continuation algorithm is closely related to the so-called smoothing Newton method (see, for example, Huang, Qi, and Sun [24] and references therein). Before we end this paper, some observations on the relationship between our results and similar results of the existing smoothing Newton methods could be helpful for future research.

- To our knowledge, all existing smoothing Newton methods require the boundedness of the solution set if global linear convergence and local superlinear convergence are considered simultaneously. Sun and Huang [34] proposed a smoothing Newton algorithm for the P_*-matrix LCP. However, the purpose of [34] is to investigate the finite termination property of the algorithm. Huang [21] proposed a smoothing Newton algorithm for the nonlinear complementarity problem, but the purpose of [21] is to investigate global convergence and other convergence behavior of the algorithm. No result on rate of convergence was reported in the two papers above.
- Generally, it is very difficult to establish global linear convergence for the smoothing Newton method. Huang, Han, and Chen [23] proposed a predictor-corrector-type smoothing Newton algorithm where they gave a global linear convergence result with respect to the reformulation function sequence $\{H(z^k)\}$, i.e., there exists a constant $c \in (0, 1)$ such that $\|H(z^{k+1})\| \leq c\|H(z^k)\|$ for all k. In this paper, under a weaker assumption than [23], we proved global linear convergence with respect to the smoothing parameter sequence $\{\mu_k\}$. Thus, the two results do not cover each other.
- Huang, Qi, and Sun [24] proposed a smoothing Newton method, which improved the local superlinear results of some existing smoothing Newton methods. In particular, the algorithm in [24] was shown to be locally superlinearly convergent under the nonsingular assumption or locally quadratically convergent under the strict comple-

mentary condition. However, such a result was established only for the monotone LCP rather than the P_*-LCP here.

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References

[1] Burke, J. and Xu, S., The global linear convergence of a non-interior path-following algorithm for linear complementarity problems, *Math. Oper. Res.*, 23 (1998), 719–734.

[2] Burke, J. and Xu, S., A non-interior predictor-corrector path following algorithm for the monotone linear complementarity problem, *Math. Program.*, 87 (2000), 113-130.

[3] Burke, J. and Xu, S., A non-interior predictor-corrector path-following algorithm for LCP, in *Reformulation: Nonsmooth, Piecewise Smooth and Smoothing Methods*, edited by Fukushima, M. and Qi, L., Kluwer Academic Publishers, Boston, pp. 45–64, 1999.

[4] Chen, B. and Chen, X., A global and local superlinear continuation-smoothing method for $P_0 + R_0$ and monotone NCP, *SIAM J. Optim.*, 9 (1999), 624–645.

[5] Chen, B. and Chen, X., A global linear and local quadratic continuation smoothing method for variational inequalities with box constraints, *Comput. Optim. Appl.*, 13 (2000), 131–158.

[6] Chen, B. and Harker, P. T., A non-interior-point continuation method for linear complementarity problem, *SIAM J. Matrix Anal. Appl.*, 14 (1993), 1168–1190.

[7] Chen, B. and Xiu, N., A global linear and local quadratic non-interior continuation method for nonlinear complementarity problems based on Chen-Mangasarian smoothing functions, *SIAM J. Optim.*, 9 (1999), 605–623.

[8] Chen, B. and Xiu, N., Superlinear noninterior one-step continuation method for monotone LCP in the absence of strict complementarity, *J. Optim. Theory Appl.*, 108 (2001), 317–332.

[9] Chen, C. and Mangasarian, O.L., A class of smoothing functions for nonlinear and mixed complementarity problems, *Comput. Optim. Appl.*, 5 (1996), 97–138.

[10] Chen, X. and Ye, Y., On smoothing methods for the P_0 matrix linear complementarity problem, *SIAM J. Optim.*, 11 (2000), 341–363.

[11] Chen, X. and Tseng, P., Non-interior continuation methods for solving semidefinite complementarity problems, *Math. Program.*, 95 (2003), 431–474.

[12] Cottle, R.W., Pang, J.-S., and Stone, R.E., *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[13] Engelke, S. and Kanzow, C., Predictor-corrector smoothing methods for the solution of linear programming, *Preprint*, Department of Mathematics, University of Hamburg, Germany, March 2000.

[14] Engelke, S. and Kanzow, C., Improved smoothing-type methods for the solution of linear programming, *Numer. Math.*, 90 (2002), 487–507.

[15] Fischer, A., A special Newton-type optimization method, *Optim.*, 24 (1992), 269–284.

[16] Gabriel, S.A. and Moré, J.J., Smoothing of mixed complementarity problems, in *Complementarity and Variational Problems: State of the Art*, edited by Ferris, M.C. and Pang, J.-S., SIAM Publishing, Philadelphia, Pennsylvania, pp.105–116, 1997.

[17] Gowda, M.S. and Tawhid, M.A., Existence and limiting behavior of trajectories associated with P_0-equations, *Comput. Optim. Appl.*, 12 (1999), 229–251.

[18] Hotta, K. and Yoshise, A., Global convergence of a non-interior-point algorithms using Chen-Harker-Kanzow functions for nonlinear complementarity problems, *Math. Program.*, 86 (1999), 105–133.

[19] Huang, Z.H., Sufficient conditions on nonemptiness and boundedness of the solution set of the P_0 function nonlinear complementarity problem, *Oper. Res. Letters*, 30 (2002), 202–210.

[20] Huang, Z.H., The global linear and local quadratic convergence of a non-interior continuation algorithm for the LCP. To appear in *IMA J. Numer. Anal.*, 2005.

[21] Huang, Z.H., Locating a maximally complementary solution of the monotone NCP by using non-interior-point smoothing algorithms. To appear in *Math. Meth. Oper. Res.*, No. 1, 2005.

[22] Huang, Z.H. and Han, J., Non-interior continuation method for solving the monotone semidefinite complementarity problem, *Appl. Math. Optim.*, 47 (2003), 195–211.

[23] Huang, Z.H., Han, J., and Chen, Z., A predictor-corrector smoothing Newton algorithm, based on a new smoothing function, for solving the nonlinear complementarity problem with a P_0 function, *J. Optim. Theory Appl.*, 117 (2003), 39–68.

[24] Huang, Z.H., Qi, L., and Sun, D., Sub-quadratic convergence of a smoothing Newton algorithm for the P_0- and monotone LCP, *Math. Program.*, 99 (2004), 423–441.

[25] Huang, Z.H. and Sun, J., A Non-Interior Continuation Algorithm for the P_0 or P_*-LCP with Strong Global and Local Convergence Properties – Detailed Draft, *Technical Report,* Dept. of Decision Sciences, National University of Singapore, 2004.

[26] Kanzow, C., Some noninterior continuation methods for linear complementarity problems, *SIAM J. Matrix Anal. Appl.*, 17 (1996), 851–868.

[27] Kanzow, C., A new approach to continuation methods for complementarity problems with P-functions, *Oper. Res. Letters*, 20 (1997), 851–868.

[28] Kojima, M., Megiddo, N., Noma, T., and Yoshise, A., A unified approach to interior point algorithms for linear complementarity problems, *Lecture Notes in Computer Sciences*, 538, Springer-Verlag, Berlin, Germany, 1991.

[29] Mifflin, R., Semismooth and semiconvex functions in constrained optimization, *SIAM J. Control Optim.*, 15 (1977), 957–972.

[30] Qi, L., Convergence analysis of some algorithm for solving nonsmooth equations. *Math. Oper. Res.*, 18 (1993), 227-244.

[31] Qi, L. and Sun, D., Improving the convergence of non-interior point algorithm for nonlinear complementarity problems, *Math. Comput.*, 69 (2000), 283–304.

[32] Qi, L. and Sun, J., A nonsmooth version of Newton's method, *Math. Program.*, 58 (1993), 353–367.

[33] Smale, S., Algorithms for solving equations, in *Proceeding of International Congress of Mathematicians*, edited by Gleason, A. M., American Mathematics Society, Providence, Rhode Island, pp.172–195, 1987.

[34] Sun, J. and Huang, Z.H., Finite termination of a smoothing Newton algorithm for solving the LCP with a sufficient matrix, *Technical Report,* Dept. of Decision Sciences, National University of Singapore, 2003.

[35] Tseng, P., Analysis of a non-interior continuation method based on Chen-Mangasarian smoothing functions for complementarity problems, in *Reformulation-Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*, edited by Fukushima, M. and Qi, L., Kluwer Academic Publishers, Boston, pp. 381–404, 1999.

[36] Tseng, P., Error bounds and superlinear convergence analysis of some Newton-type methods in optimization, in *Nonlinear Optimization and Related Topics*, edited by Di Pillo, G. and Giannessi, F., Kluwer Academic Publishers, Boston, pp. 445–462, 2000.

[37] Xu, S., The global linear convergence of an infeasible non-interior path-following algorithm for complementarity problems with uniform $P-$functions, *Math. Program.*, 87 (2000), 501–517.

[38] Zhao, Y. and Li, D., Existence and limiting behavior of a non-interior-point trajectory for nonlinear complementarity problems without strict feasibility condition, *SIAM J. Control Optim.*, 40 (2001), 898–924.

[39] Zhao, Y. and Li, D., A globally and locally superlinearly convergent non-interior-point algorithm for P_0 LCPs, *SIAM J. Optim.*, 13 (2003), 1196–1221.