Some Properties of the Augmented Lagrangian in Cone Constrained Optimization

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Abstract

A large class of optimization problems can be modeled as minimization of an objective function subject to constraints given in a form of set inclusions. We discuss in this paper augmented Lagrangian duality for such optimization problems. We formulate the augmented Lagrangian dual problems and study conditions ensuring existence of the corresponding augmented Lagrange multipliers. We also discuss sensitivity of optimal solutions to small perturbations of augmented Lagrange multipliers.

Key words: cone constraints, augmented Lagrangian, conjugate duality, duality gap, Lagrange multipliers, sensitivity analysis

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1 Introduction

The augmented Lagrangian approach to finite dimensional problems with a finite number of equality constraints was introduced in Hestenes [4] and Powell [5]. This was extended to finite dimensional inequality constrained problems by Buys [3]. Theoretical properties of the augmented Lagrangian duality method, in a finite dimensional setting with a finite number of constraints, were thoroughly investigated in Rockafellar [7].

In this paper we consider optimization problems defined in the form

$$\operatorname*{Min}_{x \in \mathcal{X}} f(x) \ \text{ subject to } \ G(x) \in K, \tag{1.1}$$

where $\mathcal{X}$ is a vector space, K is a nonempty convex subset of a vector space $\mathcal{Y}$, $f : \mathcal{X} \to \overline{\mathbb{R}}$ is an extended real valued function and $G : \mathcal{X} \to \mathcal{Y}$. We assume that $\mathcal{Y}$ is a Hilbert space equipped with a scalar product, denoted $\langle \cdot, \cdot \rangle$, and that the set K is closed in the strong (norm) topology of $\mathcal{Y}$. A large class of optimization problems can be formulated in the form (1.1). For example, in case $\mathcal{Y}$ is the linear space of $p \times p$ symmetric matrices and $K \subset \mathcal{Y}$ is the set (cone) of positive semidefinite matrices, problem (1.1) becomes a (nonlinear) semidefinite programming problem.

In the next section we introduce augmented Lagrangian dual of problem (1.1) and study its basic properties. The developments of that section follow basic ideas outlined in Rockafellar [7]. In section 3 we study the existence of augmented Lagrange multipliers. Some results of that section, and the following section 4, seem to be new even in the finite dimensional setting. In section 4 we discuss sensitivity of minimizers of the augmented Lagrangian to small perturbations of the augmented Lagrange multipliers. The analysis of sections 3 and 4 is based on a perturbation theory of optimization problems. We use [1] as a reference book for that theory.

We use the following notation and terminology throughout the paper. For a mapping $G : \mathcal{X} \to \mathcal{Y}$ we denote by $DG(x)$ its derivative at $x \in \mathcal{X}$. If $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional, we can write $DG(x)h = [\nabla G(x)]^T h$, where $\nabla G(x)$ is the Jacobian matrix of $G(\cdot)$ at x. For a set $S \subset \mathcal{Y}$ we denote by $\operatorname{int}(S)$ its interior and by

$$\sigma(y, S) := \sup_{z \in S} \langle y, z \rangle \tag{1.2}$$

its support function. The metric projection $P_K(y)$ of $y \in \mathcal{Y}$ onto the set K is defined as point $z \in K$ closest to y. That is, $P_K(y) \in K$ and $\operatorname{dist}(y, K) = \|y - P_K(y)\|$. Since $\mathcal{Y}$ is a Hilbert space and K is closed and convex, $P_K(y)$ exists and is uniquely defined. By $T_K(y)$ and $N_K(y)$ we denote the tangent and normal cones, respectively, to the set K at $y \in K$. If K is a cone, then

$$K^* := \{y \in \mathcal{Y} : \langle y, z \rangle \le 0, \ \ \forall\, z \in K\} \tag{1.3}$$

defines the (negative) dual of K. For a function $v : \mathcal{Y} \to \overline{\mathbb{R}}$ we denote by $v^*(\cdot)$ its conjugate,

$$v^*(y^*) := \sup_{y \in \mathcal{Y}} \{\langle y^*, y\rangle - v(y)\}. \tag{1.4}$$

2 Augmented duality

Consider the optimization problem (1.1), to which we refer as problem (P). The natural way of introducing augmented Lagrangian dual for this problem is by the following construction (cf., Rockafellar and Wets [10, Chapter 11, Section K*]). With problem (P) is associated the parameterized problem, denoted (P_y):

$$\operatorname*{Min}_{x \in \mathcal{X}} f(x) \ \text{ subject to } \ G(x) + y \in K. \tag{2.1}$$

Clearly, for $y = 0$, problem (P_0) coincides with problem (P). We denote by $v(y)$ the optimal value of problem (P_y), that is $v(y) := \text{val}(P_y)$. Consider the function

$$v_\tau(y) := v(y) + \tau\|y\|^2. \tag{2.2}$$

We say that $\lambda \in \mathcal{Y}$ is an *augmented Lagrange multiplier* of (P) if val(P) is finite and there exists $\tau \geq 0$ such that

$$v_\tau(y) \geq v_\tau(0) + \langle \lambda, y\rangle, \qquad \forall\, y \in \mathcal{Y} \tag{2.3}$$

(cf., [10, Example 11.62, p.524]). The above condition (2.3) means that λ is a subgradient, at $y = 0$, of the function $v_\tau(\cdot)$. The set of all λ satisfying (2.3) is called the subdifferential of $v_\tau(y)$, at $y = 0$, and denoted $\partial v_\tau(0)$. Note that $\partial v_\tau(0)$ is defined only if $v_\tau(0) = \text{val}(P)$ is finite.

We denote by $\mathcal{A}$ the set of all augmented Lagrange multipliers. Since $v_\tau(0) = v(0)$, it immediately follows from the definition that if $\tau \leq \tau'$, then $\partial v_\tau(0) \subset \partial v_{\tau'}(0)$, and hence

$$\mathcal{A} = \cup_{\tau \in \mathbb{R}_+} \partial v_\tau(0). \tag{2.4}$$

It follows that the set $\mathcal{A}$ is convex. Note also that if $v(\cdot)$ is convex, then $\partial v_\tau(0)$ coincides with $\partial v(0)$ for any $\tau \geq 0$, and hence in that case $\mathcal{A} = \partial v(0)$.

The function

$$\mathcal{L}(x, \lambda, \tau) := \inf_{y \in K - G(x)} \left\{ f(x) - \langle \lambda, y\rangle + \tau\|y\|^2 \right\} \tag{2.5}$$

is called the *augmented Lagrangian* of problem (P). We have that

$$\begin{aligned} \inf_{y \in \mathcal{Y}} \{v_\tau(y) - \langle \lambda, y\rangle\} &= \inf_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \left\{ f(x) - \langle \lambda, y\rangle + \tau\|y\|^2 : G(x) + y \in K \right\} \\ &= \inf_{x \in \mathcal{X}} \inf_{y \in K - G(x)} \left\{ f(x) - \langle \lambda, y\rangle + \tau\|y\|^2 \right\}, \end{aligned}$$

and hence (cf., [10, p.519])

$$\inf_{y\in\mathcal{Y}} \{v_\tau(y) - \langle\lambda, y\rangle\} = \inf_{x\in\mathcal{X}} \mathcal{L}(x,\lambda,\tau). \tag{2.6}$$

It is straightforward to verify that for $\tau > 0$ the augmented Lagrangian can be written in the form (cf., [10, p.521]):

$$\mathcal{L}(x,\lambda,\tau) = f(x) + \tau[\,\mathrm{dist}\,(G(x) + (2\tau)^{-1}\lambda, K)]^2 - (4\tau)^{-1}\|\lambda\|^2, \tag{2.7}$$

while $\mathcal{L}(x,\lambda,0) = L(x,\lambda) - \sigma(\lambda,K)$ for $\tau = 0$. Here

$$L(x,\lambda) := f(x) + \langle\lambda, G(x)\rangle \tag{2.8}$$

is the (standard) Lagrangian of problem (P). It follows from (2.5) and (2.6), respectively, that the functions $\mathcal{L}(x,\cdot,\cdot)$ and

$$g(\lambda,\tau) := \inf_{x\in\mathcal{X}} \mathcal{L}(x,\lambda,\tau)$$

are concave and upper semicontinuous on $\mathcal{Y}\times\mathbb{R}$.

Consider the metric projection operator $P_K(\cdot)$ onto the set K. If K is a convex *cone*, then $\mathrm{dist}(y,K) = \|P_{K^*}(y)\|$. Consequently, in that case for $\tau > 0$ the augmented Lagrangian can be written as follows

$$\mathcal{L}(x,\lambda,\tau) = f(x) + \frac{1}{4\tau}\left(\|P_{K^*}(\lambda + 2\tau G(x))\|^2 - \|\lambda\|^2\right), \tag{2.9}$$

while for $\tau = 0$ it is given by $\mathcal{L}(x,\lambda,0) = L(x,\lambda)$ if $\lambda\in K^*$ and $\mathcal{L}(x,\lambda,0) = -\infty$ if $\lambda\notin K^*$.

In the finite dimensional setting the following result is well known (cf., [9, p.212]), we give its proof for the sake of completeness.

Lemma 2.1 *For any $\tau \geq 0$ we have that*

$$\sup_{\lambda\in\mathcal{Y}} \mathcal{L}(x,\lambda,\tau) = \begin{cases} f(x), & \text{if } G(x)\in K, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.10}$$

and

$$\mathrm{val}(P) = \inf_{x\in\mathcal{X}} \sup_{\lambda\in\mathcal{Y}} \mathcal{L}(x,\lambda,\tau). \tag{2.11}$$

Proof. Suppose first that $\tau > 0$. Then

$$[\,\mathrm{dist}\,(G(x) + (2\tau)^{-1}\lambda, K)]^2 - \|(2\tau)^{-1}\lambda\|^2 = \mathrm{dist}\,(\mu, S)^2 - \|\mu\|^2, \tag{2.12}$$

where $\mu := (2\tau)^{-1}\lambda$ and $S := K - G(x)$. If $0 \in S$, i.e., $G(x) \in K$, then the right hand side of (2.12) is less than or equal to zero, and hence in that case the supremum of the right hand side of (2.12), over $\mu \in \mathcal{Y}$, is zero. On the other hand if $0 \notin S$, then for $\mu := -aP_S(0)$, with $a > 0$, we have $\text{dist}(\mu, S) = (1+a)\|P_S(0)\|$. By letting $a \to +\infty$, we obtain that the supremum of the right hand side of (2.12) is $+\infty$, and hence (2.10) follows. For $\tau = 0$ the proof of (2.10) is similar.

Equation (2.11) follows immediately from (2.10). ∎

For $\tau \geq 0$, consider the following dual, denoted (D_τ), of the problem (P):

$$\operatorname*{Max}_{\lambda \in \mathcal{Y}} \left\{ g(\lambda, \tau) = \inf_{x \in \mathcal{X}} \mathcal{L}(x, \lambda, \tau) \right\}. \tag{2.13}$$

Let us make the following observations. The equation (2.6) can be written in the form

$$g(\lambda, \tau) = -v_\tau^*(\lambda), \tag{2.14}$$

where $v_\tau^*(\cdot)$ is the conjugate of the function $v_\tau(\cdot)$. Therefore, we obtain that

$$\text{val}(D_\tau) = v_\tau^{**}(0). \tag{2.15}$$

Recall that by the Fenchel-Moreau Theorem (see, e.g., [8]) we have that

$$v_\tau^{**} = \text{cl}(\text{conv}\, v_\tau). \tag{2.16}$$

The results presented in the following theorem are quite standard in the conjugate duality theory (cf., [7],[9, p.213]).

Theorem 2.1 *For any $\tau \geq 0$ the following holds:* (i) $\text{val}(P) \geq \text{val}(D_\tau)$ *and*

$$\text{val}(D_\tau) = \text{cl}(\text{conv}\, v_\tau)(0), \tag{2.17}$$

(ii) $\text{val}(P) = \text{val}(D_\tau)$ *if and only if* $\text{cl}(\text{conv}\, v_\tau)(0) = v_\tau(0)$, (iii) *if $\partial v_\tau(0)$ is nonempty, then* $\text{val}(P) = \text{val}(D_\tau)$ *and the set of optimal solutions of (D_τ) coincides with $\partial v_\tau(0)$,* (iv) *if* $\text{val}(P) = \text{val}(D_\tau)$ *and is finite, then the (possibly empty) set of optimal solutions of (D_τ) coincides with $\partial v_\tau(0)$,* (v) $\text{val}(P) = \text{val}(D_\tau)$ *and $\bar{x}$ and $\bar{\lambda}$ are optimal solutions of (P) and (D_τ), respectively, if and only if $(\bar{x}, \bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot, \cdot, \tau)$, i.e.,*

$$\mathcal{L}(\bar{x}, \lambda, \tau) \leq \mathcal{L}(\bar{x}, \bar{\lambda}, \tau) \leq \mathcal{L}(x, \bar{\lambda}, \tau), \quad \forall\, (x, \lambda) \in \mathcal{X} \times \mathcal{Y}. \tag{2.18}$$

Proof. The inequality $\text{val}(P) \geq \text{val}(D_\tau)$ follows from the min-max representations (2.11) and (2.13), and (2.17) is a consequence of (2.15) and (2.16). Property (ii) is a consequence of (2.17). Properties (iii) and (iv) follow by general duality theory (e.g., [1, Theorem 2.142]). Property (v) follows from (2.11) and (2.13). ∎

It is also clear that the optimal value in the left hand side of (2.6), and hence $g(\lambda,\tau)$ and $\mathrm{val}(D_\tau)$, are monotonically nondecreasing with increase of τ. It follows that if $\mathrm{val}(P)=\mathrm{val}(D_\tau)$ for some $\tau\ge 0$, then $\mathrm{val}(P)=\mathrm{val}(D_{\tau'})$ for any $\tau'\ge\tau$. As a consequence of the above results we obtain the following.

Theorem 2.2 *If for some $\bar\tau\ge 0$ the set $\partial v_{\bar\tau}(0)$ is nonempty (i.e., there exists an augmented Lagrange multiplier), then for any $\tau\ge\bar\tau$ there is no duality gap between problems (P) and (D_τ), and for any $\bar\lambda\in\partial v_\tau(0)$ the (possibly empty) set of optimal solutions of (P) is contained in the set $\arg\min_{x\in\mathcal{X}}\mathcal{L}(x,\bar\lambda,\tau)$.*

The following condition was introduced in Rockafellar [7] and called there: "the quadratic growth condition". Since we use the term "quadratic growth" for a different meaning, we refer to it as Condition (R).

Condition (R) There exist constants a and b such that:

$$v(y)\ge a-b\|y\|^2,\quad \forall\, y\in\mathcal{Y}. \tag{2.19}$$

If, for example, $f(x)$ is bounded from below on $\mathcal{X}$ by a constant c, then (2.19) holds with $a:=c$ and $b:=0$.

Lemma 2.2 *Suppose that $\liminf_{y\to 0}v(y)<+\infty$ and condition (R) holds. Then for any $\lambda\in\mathcal{Y}$,*

$$\lim_{\tau\to+\infty}g(\lambda,\tau)=\liminf_{y\to 0}v(y). \tag{2.20}$$

Proof. By (2.6) we have

$$g(\lambda,\tau)=\inf_{y\in\mathcal{Y}}\left\{v(y)+\tau\|y\|^2-\langle\lambda,y\rangle\right\}. \tag{2.21}$$

It follows that for any τ, $g(\lambda,\tau)$ is less than or equal to the right hand side of (2.20), and hence

$$\lim_{\tau\to+\infty}g(\lambda,\tau)\le\liminf_{y\to 0}v(y).$$

Now it follows from condition (R) that, for any $y\in\mathcal{Y}$,

$$v_\tau(y)-\langle\lambda,y\rangle\ge a-\|\lambda\|\,\|y\|+(\tau-b)\|y\|^2, \tag{2.22}$$

and hence $\liminf_{y\to 0}v(y)>-\infty$. It follows that $\liminf_{y\to 0}v(y)$ is finite. We have then that for any $r>0$ there exists τ^* such that for $\tau\ge\tau^*$ the infimum of the right hand side of (2.22), over all y satisfying $\|y\| > r$, is bigger than $\liminf_{y\to 0} v(y)$. Consequently, by (2.22) we obtain that for any $r > 0$,

$$\sup_{\tau\in\mathbb{R}_+} \inf_{\|y\|>r} \{v_\tau(y) - \langle\lambda, y\rangle\} \geq \liminf_{y\to 0} v(y) = \liminf_{y\to 0} [v_\tau(y) - \langle\lambda, y\rangle].$$

Together with (2.21) this implies that

$$\lim_{\tau\to+\infty} g(\lambda,\tau) \geq \liminf_{y\to 0} v(y),$$

and hence (2.20) follows. ∎

Consider the following problem, denoted $(\widehat{D})$,

$$\operatorname*{Max}_{(\lambda,\tau)\in\mathcal{Y}\times\mathbb{R}_+} g(\lambda,\tau). \tag{2.23}$$

Clearly $\mathrm{val}(\widehat{D}) = \sup_{\tau\in\mathbb{R}_+} \mathrm{val}(D_\tau)$. Since for any $\tau \geq 0$ we have that $\mathrm{val}(D_\tau) \leq \mathrm{val}(P)$, it follows that $\mathrm{val}(\widehat{D}) \leq \mathrm{val}(P)$.

Theorem 2.3 *Suppose that* $\liminf_{y\to 0} v(y) < +\infty$ *and condition* (R) *holds. Then*

$$\mathrm{val}(\widehat{D}) = \liminf_{y\to 0} v(y) \leq \mathrm{val}(P). \tag{2.24}$$

Proof. By Lemma 2.2 we have that

$$\mathrm{val}(\widehat{D}) \geq \liminf_{y\to 0} v(y). \tag{2.25}$$

Because of (2.17) we have that for any $\tau \geq 0$,

$$\mathrm{val}(D_\tau) \leq \liminf_{y\to 0} v_\tau(y) = \liminf_{y\to 0} v(y),$$

and hence the opposite of inequality (2.25) holds. This proves the first equality statement in (2.24). The inequality statement of (2.24) follows from $\mathrm{val}(\widehat{D}) \leq \mathrm{val}(P)$. ∎

It is said that there is *no duality gap* between problems $(\widehat{D})$ and (P) if

$$\mathrm{val}(\widehat{D}) = \mathrm{val}(P). \tag{2.26}$$

It follows from (2.24) that the "no duality gap" condition (2.26) holds iff the optimal value function $v(y)$ is lower semicontinuous at $y = 0$. There exist various conditions ensuring lower semicontinuity of $v(y)$ at $y = 0$. One such condition is that the space $\mathcal{X}$ is a topological vector space, the function $f(\cdot)$ is lower semicontinuous, the mapping $G(\cdot)$ is continuous, and the so-called inf-compactness condition holds (e.g., [1, Proposition 4.4]).

Theorem 2.4 *Suppose that condition* (R) *is satisfied,* val(P) *is finite and the optimal value function* $v(y)$ *is lower semicontinuous at* $y = 0$*. Then the following holds:* (i) *there is no duality gap between* $(\widehat{D})$ *and* (P)*,* (ii) *if* $(\bar{\lambda}, \bar{\tau})$ *is an optimal solution of* $(\widehat{D})$*, then* $\text{val}(D_{\bar{\tau}}) = \text{val}(P)$ *and* $\bar{\lambda} \in \partial v_{\bar{\tau}}(0)$*,* (iii) *if* $\bar{\lambda} \in \partial v_{\bar{\tau}}(0)$*, then* $(\bar{\lambda}, \bar{\tau})$ *is an optimal solution of* $(\widehat{D})$*.*

Proof. The no duality gap property (i) was already discussed above. Now if $(\bar{\lambda}, \bar{\tau})$ is an optimal solution of $(\widehat{D})$, then, since $\text{val}(D_\tau)$ is nondecreasing as a function τ, we have that $\text{val}(D_{\bar{\tau}}) = \text{val}(\widehat{D})$. It follows that $\text{val}(D_{\bar{\tau}}) = \text{val}(P)$ and $\bar{\lambda} \in \partial v_{\bar{\tau}}(0)$. Conversely, if this holds, then again by the monotonicity we obtain that $\text{val}(D_{\bar{\tau}}) = \text{val}(P)$ and $(\bar{\lambda}, \bar{\tau})$ is an optimal solution of $(\widehat{D})$. ∎

3 Existence of augmented Lagrange multipliers

In this section we discuss existence of augmented Lagrange multipliers. We assume that condition (R) holds and val(P) is finite. The following lemma shows that in this case verification of condition (2.3) can be reduced to a local analysis. Denote $B_r := \{y : \|y\| \le r\}$.

Lemma 3.1 *Suppose that condition* (R) *holds and* val(P) *is finite. Then* λ *is an augmented Lagrange multiplier if and only if for any* $\varepsilon > 0$ *there exists* $\tau \ge 0$ *such that*

$$v_\tau(y) \ge v_\tau(0) + \langle \lambda, y \rangle, \qquad \forall\, y \in B_\varepsilon. \tag{3.1}$$

Proof. Necessity of condition (3.1) follows directly from the definition. Let us show sufficiency. It follows from condition (R) that

$$v_\tau(y) - v_\tau(0) - \langle \lambda, y \rangle \ge a - v(0) - \|\lambda\|\,\|y\| + (\tau - b)\|y\|^2. \tag{3.2}$$

This implies that, for $\tau > b$, the left hand side of (3.2) is nonnegative for all y such that $\|y\| \ge r(\tau)$, where

$$r(\tau) := \frac{\|\lambda\|}{2(\tau - b)} + \sqrt{\frac{\|\lambda\|^2}{4(\tau - b)^2} + \frac{v(0) - a}{\tau - b}}. \tag{3.3}$$

Therefore, λ is an augmented Lagrange multiplier if, for $\tau > b$, the inequality in (2.3) holds for all $y \in B_r$, where $r = r(\tau)$ is defined in (3.3). Clearly, for a fixed λ, $r(\tau) \to 0$ as $\tau \to +\infty$. Now if condition (3.1) holds for some $\tau \ge 0$ and $\varepsilon > 0$, then it holds for any bigger value of τ and the same ε. By taking τ large enough so that $\tau > b$ and $\varepsilon > r(\tau)$, we obtain that (3.1) implies condition (2.3) for all $y \in \mathcal{Y}$, and hence λ is an augmented Lagrange multiplier. ∎

We assume now that the space $\mathcal{X}$ is a Banach space, the function $f : \mathcal{X} \to \mathbb{R}$ is real valued and $f(\cdot)$ and $G(\cdot)$ are continuously differentiable. Let x_0 be an optimal solution of the problem (P). Existence of such an optimal solution implies, of course, that $\mathrm{val}(P)$ is finite. We denote by $\Lambda(x_0)$ the set of Lagrange multipliers satisfying the first order optimality conditions at the point x_0:

$$D_x L(x_0, \lambda) = 0, \ \ \lambda \in N_K(G(x_0)). \tag{3.4}$$

Consider the function $\delta(\cdot) := \mathrm{dist}(\cdot, K)^2$. We have that

$$\delta(y) = \inf_{z \in K} \|y - z\|^2, \tag{3.5}$$

and hence the function $\delta(\cdot)$ is convex and differentiable with

$$D\delta(y) = 2(y - P_K(y)), \tag{3.6}$$

where (3.6) follows, for example, by Danskin Theorem (e.g., [1, p. 273]). Moreover, $P_K(\cdot)$ is Lipschitz continuous (modulus one), and hence $D\delta(\cdot)$ is Lipschitz continuous. By the chain rule of differentiation we obtain that

$$D_\lambda \mathcal{L}(x, \lambda, \tau) = G(x) - P_K\left((G(x) + (2\tau)^{-1}\lambda\right). \tag{3.7}$$

Let $\bar\lambda$ be an augmented Lagrange multiplier, i.e., $\bar\lambda \in \partial v_\tau(0)$ for some $\tau \geq 0$. Recall that existence of the augmented Lagrange multiplier $\bar\lambda$ implies that $\mathrm{val}(P) = \mathrm{val}(D_\tau)$. Suppose also that the problem (P) has an optimal solution x_0. Then $(x_0, \bar\lambda)$ is a saddle point of $\mathcal{L}(\cdot, \cdot, \tau)$, and hence $D_\lambda \mathcal{L}(x_0, \bar\lambda, \tau) = 0$ and $D_x \mathcal{L}(x_0, \bar\lambda, \tau) = 0$. By (3.7) the first of these equations means that

$$G(x_0) = P_K\left((G(x_0) + (2\tau)^{-1}\bar\lambda\right). \tag{3.8}$$

We also have that $y - P_K(y) \in N_K(P_K(y))$ for any $y \in \mathcal{Y}$. By applying this to $y := G(x_0) + (2\tau)^{-1}\bar\lambda$ and using (3.8) we obtain that $\bar\lambda \in N_K(G(x_0))$. By using (3.6) and (3.8) it is also straightforward to verify that

$$D_x \mathcal{L}(x_0, \bar\lambda, \tau) = D_x L(x_0, \bar\lambda), \tag{3.9}$$

and hence $D_x L(x_0, \bar\lambda) = 0$. It follows that $\bar\lambda \in \Lambda(x_0)$. We obtain the following result.

Proposition 3.1 *If x_0 is an optimal solution of the problem (P), then $\mathcal{A} \subset \Lambda(x_0)$.*

It may be interesting to remark that it follows from the above proposition that if S^* is the set of optimal solutions of (P), then $\mathcal{A} \subset \cap_{x \in S^*} \Lambda(x)$. In particular, if there are two points in S^* with disjoint sets of Lagrange multipliers, then problem (P) does not possess augmented Lagrange multipliers.

Let us consider now the lower directional derivative of $v(\cdot)$ at $y = 0$, defined as

$$v'_-(0, d) := \liminf_{t \downarrow 0} \frac{v(td) - v(0)}{t}. \tag{3.10}$$

The upper directional derivative $v'_+(0, d)$ is defined similarly by taking lim inf instead of lim sup in (3.10). It is said that $v(y)$ is directionally differentiable at $y = 0$ if $v'_-(0, d) = v'_+(0, d)$ for all $d \in \mathcal{Y}$. In that case the directional derivative $v'(0, d)$ is equal to $v'_-(0, d)$ and $v'_+(0, d)$.

Theorem 3.1 *Let x_0 be an optimal solution of the problem (P). Suppose that Robinson's constraint qualification holds at x_0, and $\mathcal{A} = \Lambda(x_0)$. Then $v(\cdot)$ is directionally differentiable at $y = 0$ and $v'(0, d) = \sigma(d, \Lambda(x_0))$.*

Proof. We have that for any $\lambda \in \partial v_\tau(0)$,

$$v(td) - v(0) \geq -t^2 \tau \|d\|^2 + t\langle \lambda, d \rangle,$$

and hence $v'_-(0, d) \geq \langle \lambda, d \rangle$. It follows that if the set $\mathcal{A}$ of augmented Lagrange multipliers is nonempty, then

$$v'_-(0, d) \geq \sigma(d, \mathcal{A}), \quad \forall\, d \in \mathcal{Y}. \tag{3.11}$$

Suppose that Robinson's constraint qualification holds at x_0. Then the set $\Lambda(x_0)$ is nonempty and bounded (e.g., [1, Theorem 3.9]) and

$$v'_+(0, d) \leq \sigma(d, \Lambda(x_0)), \quad \forall\, d \in \mathcal{Y}, \tag{3.12}$$

(see [1, Proposition 4.22]). By (3.11) and (3.12) we have that for any $d \in \mathcal{Y}$,

$$\sigma(d, \mathcal{A}) \leq v'_-(0, d) \leq v'_+(0, d) \leq \sigma(d, \Lambda(x_0)).$$

If, moreover, $\mathcal{A} = \Lambda(x_0)$, then $v'_-(0, d) = v'_+(0, d)$ and is equal to $\sigma(d, \Lambda(x_0))$. ∎

Let us observe at this point that even if (P) has unique optimal solution x_0 at which Robinson's constraint qualification holds, it still may happen that $v'_+(0, d)$ is strictly less than $\sigma(d, \Lambda(x_0))$ for some $d \in \mathcal{Y}$ (see [1, Proposition 4.108] and the following discussion). In such a case the inclusion $\mathcal{A} \subset \Lambda(x_0)$ is strict. A simple example of a finite dimensional problem (3 decision variables and 3 constraints) with unique optimal solution, satisfying a weak second order optimality condition, and having a nonempty and bounded set of Lagrange multipliers, and yet with empty set of augmented Lagrange multipliers, is given in [2, section 14.3].

Let x_0 be an optimal solution of the problem (P) and suppose that Robinson's constraint qualification holds at x_0. If, further, $\mathcal{A} = \Lambda(x_0)$, then by Theorem 3.1 we have that $v'(0, \cdot) = \sigma(\cdot, \Lambda(x_0))$. This, in turn, implies that for any $d \in \mathcal{Y}$ and $\varepsilon > 0$ there exists a $(t\varepsilon)$-optimal $(t \geq 0)$ solution $\bar{x}(t)$ of (P_{td}) such that $\|\bar{x}(t) - x_0\| = O(t)$ (see [1, p.282]). Therefore, property $\mathcal{A} = \Lambda(x_0)$ is closely related to Lipschitz stability of optimal (nearly optimal) solutions of the parameterized problem (P_y).

Theorem 3.2 *Let x_0 be an optimal solution of the problem (P). Suppose that:* (i) *condition* (R) *holds,* (ii) *$f(\cdot)$ and $G(\cdot)$ are $C^{1,1}$ (i.e., $f(\cdot)$ and $G(\cdot)$ are differentiable and their derivatives are locally Lipschitz continuous),* (iii) *Robinson's constraint qualification holds at x_0,* (iv) *for all y in a neighborhood of $0 \in \mathcal{Y}$, problem (P_y) has an $\varepsilon(y)$–optimal solution $\bar{x}(y)$ such that $\varepsilon(y) = O(\|y\|^2)$ and*

$$\|\bar{x}(y) - x_0\| = O(\|y\|). \tag{3.13}$$

Then $\mathcal{A} = \Lambda(x_0)$.

Proof. By Lemma 3.1 we need to verify the corresponding subgradient inequality only for all y near $0 \in \mathcal{Y}$. We have that

$$v(y) - v(0) = f(\bar{x}(y)) - f(x_0) + O(\|y\|^2). \tag{3.14}$$

Moreover, for any $\lambda \in \Lambda(x_0)$ we have that $\lambda \in N_K(G(x_0))$, and since $G(x_0) \in K$ and $G(\bar{x}(y)) + y \in K$ it follows that

$$\langle \lambda, G(\bar{x}(y)) + y - G(x_0) \rangle \leq 0.$$

Consequently

$$f(\bar{x}(y)) - f(x_0) \geq L(\bar{x}(y), \lambda) - L(x_0, \lambda) + \langle \lambda, y \rangle. \tag{3.15}$$

Since f and G are $C^{1,1}$, and by the first order optimality conditions, $D_x L(x_0, \lambda) = 0$, and because of the assumption $\|\bar{x}(y) - x_0\| = O(\|y\|)$, we have that

$$|L(\bar{x}(y), \lambda) - L(x_0, \lambda)| = O(\|y\|^2). \tag{3.16}$$

It follows from (3.14)–(3.16) that for any $\lambda \in \Lambda(x_0)$,

$$v(y) - v(0) \geq \langle \lambda, y \rangle + O(\|y\|^2). \tag{3.17}$$

This implies (3.1) for some $\varepsilon > 0$ and $\tau \geq 0$ large enough, and hence $\lambda \in \mathcal{A}$. We obtain that $\Lambda(x_0) \subset \mathcal{A}$, which together with the opposite inclusion (see Proposition

3.1) imply that $\Lambda(x_0) = \mathcal{A}$. ∎

From the assumptions (i)–(iv) of the above theorem, the last one is the most delicate, of course. In order to verify this condition one can apply results from the theory of parametric optimization about Lipschitz stability of optimal (nearly optimal) solutions (see, e.g., [1, Chapter 4]). It is known that Lipschitz stability of optimal solutions is closely related the second order properties of the problem (P). We present a second analysis in the next section.

3.1 Second order conditions

In this section we discuss second order optimality conditions ensuring existence of augmented Lagrange multipliers. We assume throughout this section that the spaces $\mathcal{X}$ and $\mathcal{Y}$ are *finite* dimensional, and the function $f : \mathcal{X} \to \mathbb{R}$ and the mapping $G : \mathcal{X} \to \mathcal{Y}$ are twice continuously differentiable.

Let $x_0 \in \mathcal{X}$ be a stationary point of the problem (P), i.e., the set $\Lambda(x_0)$ of Lagrange multipliers, satisfying the first order necessary conditions (3.4), is nonempty. Let $\bar{\lambda} \in \Lambda(x_0)$ and consider the function $\ell(x) := \mathcal{L}(x, \bar{\lambda}, \bar{\tau})$ for some $\bar{\tau} \geq 0$. Note that $D\ell(x_0) = D_x L(x_0, \bar{\lambda}) = 0$ by the first order necessary conditions (compare with derivations of (3.7)–(3.9)). We say that the *quadratic growth* condition, for the function $\ell(x)$, holds at x_0 if there exist a neighborhood $\mathcal{N} \subset \mathcal{X}$ of x_0 and constant $c > 0$ such that

$$\ell(x) \geq \ell(x_0) + c\|x - x_0\|^2, \quad \forall\, x \in \mathcal{N}. \tag{3.18}$$

Suppose that the set K is *second order regular* (see [1, section 3.3.3] for a definition and discussion of the concept of second order regularity). We have then the following result ([1, Theorem 4.133]): for any $y, d \in \mathcal{Y}$ it holds that

$$\lim_{\substack{t \downarrow 0 \\ d' \to d}} \frac{\delta(y + td') - \delta(y) - tD\delta(y)d'}{\frac{1}{2}t^2} = \nu(d), \tag{3.19}$$

where $\nu(d)$ is the optimal value of the problem

$$\operatorname*{Min}_{z \in \mathcal{C}(y)} \left\{ 2\|d - z\|^2 - 2\sigma(y - \bar{y}, T_K^2(\bar{y}, z)) \right\}, \tag{3.20}$$

$\bar{y} := P_K(y)$ and $\mathcal{C}(y) := \{z \in T_K(\bar{y}) : \langle y - \bar{y}, z \rangle = 0\}$. It follows that the function $\delta(\cdot)$ is second order epiregular (see [1, section 3.3.4] for a discussion of the concept of second order epiregularity) and

$$\delta''(y; d, r) = D\delta(y)r + \nu(d) \tag{3.21}$$

(see [1, Remark 4.134]).

It follows by the second order epiregularity of $\delta(\cdot)$ that the function $\ell(\cdot)$ is also second order epiregular, and by a chain rule (see [1, p. 44])

$$\begin{array}{l} \ell''(x_0; h, w) = Df(x_0)w + D^2 f(x_0)(h,h) \\ \quad +\bar{\tau}\delta''(G(x_0) + (2\bar{\tau})^{-1}\bar{\lambda}; DG(x_0)h, DG(x_0)w + D^2G(x_0)(h,h)). \end{array} \tag{3.22}$$

By using (3.6) and (3.21), and since $P_K(G(x_0) + (2\tau)^{-1}\bar{\lambda}) = G(x_0)$, for $\tau \geq 0$, and $D_x L(x_0, \bar{\lambda}) = 0$, we obtain that

$$\ell''(x_0; h, w) = D^2_{xx} L(x_0, \bar{\lambda})(h,h) + \vartheta_{\bar{\tau}}(h), \tag{3.23}$$

where $\vartheta_{\bar{\tau}}(h)$ is the optimal value of the problem:

$$\operatorname*{Min}_{z \in \mathcal{C}(x_0)} \left\{ 2\bar{\tau}\|DG(x_0)h - z\|^2 - \sigma(\bar{\lambda}, T^2_K(G(x_0), z) \right\}, \tag{3.24}$$

with

$$\mathcal{C}(x_0) := \left\{ z \in T_K(G(x_0)) : \langle \bar{\lambda}, z \rangle = 0 \right\}. \tag{3.25}$$

If x_0 is a local minimizer of $\ell(\cdot)$, then the following second order necessary condition holds

$$\inf_{w \in \mathcal{X}} \ell''(x_0; h, w) \geq 0, \quad \forall\, h \in \mathcal{X}. \tag{3.26}$$

Also, because of the second order epiregularity of $\ell(\cdot)$, the second order growth condition (3.18) holds iff

$$\inf_{w \in \mathcal{X}} \ell''(x_0; h, w) > 0, \quad \forall\, h \in \mathcal{X} \setminus \{0\}, \tag{3.27}$$

[1, Proposition 3.105]. By (3.23) we have that $\ell''(x_0; h, w)$ does not depend on w. Therefore we obtain the following results.

Theorem 3.3 *Let $\bar{\lambda} \in \Lambda(x_0)$ and suppose that the set K is second order regular. Then the following holds:* (i) *if x_0 is a local minimizer of of $\mathcal{L}(\cdot, \bar{\lambda}, \bar{\tau})$, then necessarily*

$$D^2_{xx} L(x_0, \bar{\lambda})(h,h) + \vartheta_{\bar{\tau}}(h) \geq 0, \quad \forall\, h \in \mathcal{X}, \tag{3.28}$$

(ii) *the quadratic growth condition* (3.18) *holds if and only if the following condition is satisfied:*

$$D^2_{xx} L(x_0, \bar{\lambda})(h,h) + \vartheta_{\bar{\tau}}(h) > 0, \quad \forall\, h \in \mathcal{X} \setminus \{0\}, \tag{3.29}$$

where $\vartheta_{\bar{\tau}}(h)$ is the optimal value of the problem (3.24).

We can compare the second order conditions (3.28) and (3.29) with the corresponding second order optimality conditions for the problem (P). Consider the critical cone

$$C(x_0) := \big\{h \in \mathcal{X} : DG(x_0)h \in T_K(G(x_0)),\ \langle \bar\lambda, DG(x_0)h\rangle = 0\big\} \tag{3.30}$$

of the problem (P). We have that if $h \in C(x_0)$, then by taking $z := DG(x_0)h$ in (3.24) we obtain that

$$\vartheta_{\bar\tau}(h) \leq -\sigma(\bar\lambda, T_K^2(G(x_0), DG(x_0)h)).$$

Therefore, condition (3.29) implies

$$D^2_{xx}L(x_0,\bar\lambda)(h,h) - \sigma(\bar\lambda, T_K^2(G(x_0), DG(x_0)h)) > 0, \quad \forall\, h \in C(x_0)\setminus\{0\}. \tag{3.31}$$

Since $\bar\lambda \in \Lambda(x_0)$ and because of the second order regularity of K, condition (3.31) in turn implies the quadratic growth condition for the problem (P) at the point x_0 (cf., [1, Theorem 3.86]).

Since K is convex, we have that the function

$$\phi(\cdot) := -\sigma(\bar\lambda, T_K^2(G(x_0), \cdot)) \tag{3.32}$$

is convex ([1, Proposition 3.48]) and, since $\bar\lambda \in \Lambda(x_0)$, that

$$-\sigma(\bar\lambda, T_K^2(G(x_0), z)) \geq 0, \ \ \forall\, z \in \mathcal{C}(x_0) \tag{3.33}$$

(e.g., [1, p.178]). Note also that the function $\phi(\cdot)$, and hence the function $\vartheta_{\bar\tau}(\cdot)$, is second order positively homogeneous, i.e., $\phi(th) = t^2\phi(h)$ for any $t \geq 0$ and h.

Theorem 3.4 *Let x_0 be an optimal solution of the problem (P) and $\bar\lambda \in \Lambda(x_0)$. Suppose that the set K is second order regular. Then the following holds.* (a) *If $\bar\lambda$ is an augmented Lagrange multiplier, then condition* (3.28) *holds for some $\bar\tau \geq 0$.* (b) *Conversely, suppose (in addition) that:* (i) *condition* (R) *holds,* (ii) *the second order condition* (3.31) *is satisfied,* (iii) *for all y in a neighborhood of $0 \in \mathcal{Y}$ the problem (P_y) has an optimal solution $\bar x(y)$ converging to x_0 as $y \to 0$,* (iv) *the function $\phi(\cdot)$ is lower semicontinuous on the set $\mathcal{C}(x_0)$. Then $\bar\lambda$ is an augmented Lagrange multiplier.*

Proof. Suppose that $\bar\lambda$ is an augmented Lagrange multiplier, i.e., $\bar\lambda \in \partial v_{\bar\tau}(0)$ for some $\bar\tau \geq 0$. Then $(x_0, \bar\lambda)$ is a saddle point of $\mathcal{L}(\cdot,\cdot,\bar\tau)$ (see Theorem 2.1), and hence x_0 is a minimizer of $\mathcal{L}(\cdot,\bar\lambda,\bar\tau)$. Consequently, the necessary condition (3.28) follows. This completes the proof of property (a).

In order to prove (b) we need to show that $(x_0, \bar\lambda)$ is a saddle point of $\mathcal{L}(\cdot,\cdot,\hat\tau)$ for some $\hat\tau \geq 0$. We have that for any $\tau \geq 0$, the function $\mathcal{L}(x_0,\cdot,\tau)$ is concave and

$D_\lambda \mathcal{L}(x_0, \bar{\lambda}, \tau) = 0$. It follows that $\mathcal{L}(x_0, \lambda, \tau) \le \mathcal{L}(x_0, \bar{\lambda}, \tau)$ for any $\lambda \in \mathcal{Y}$. Now by (2.6) and Lemma 2.2 we have, for $\varepsilon > 0$ and $\tau \ge 0$ large enough,

$$\inf_{x \in \mathcal{X}} \mathcal{L}(x, \bar{\lambda}, \tau) = \inf_{y \in B_\varepsilon} \left\{ f(\bar{x}(y)) + \tau \|y\|^2 - \langle \bar{\lambda}, y \rangle \right\}.$$

Consequently, it suffices to verify that $\mathcal{L}(x, \bar{\lambda}, \tau) \ge \mathcal{L}(x_0, \bar{\lambda}, \tau)$ only for all x in a neighborhood of x_0. Therefore, it only remains to show that condition (3.31) implies condition (3.29) for some $\bar{\tau} \ge 0$.

Suppose that condition (3.31) holds. Because of the second order regularity of the set K, we have that the second order tangent set $T_K^2(G(x_0), z)$ is nonempty for all $z \in \mathcal{C}(x_0)$. Together with (3.33) this implies that the function $\phi(z)$, defined in (3.32), is finite valued for all $z \in \mathcal{C}(x_0)$. It follows then that the number

$$\alpha := \sup \{ \phi(DG(x_0)h) : h \in C(x_0),\ \|h\| \le 1 \}$$

is finite. Indeed, by convexity of $\phi(DG(x_0)\cdot)$ we have that for any $h \in C(x_0)$,

$$\phi(DG(x_0)h) \ge \liminf_{h' \to h} \phi(DG(x_0)h')$$

(e.g., [1, pp. 76-77]). Since the set $S := \{h \in C(x_0),\ \|h\| \le 1\}$ is compact and $\phi(DG(x_0)h)$ is finite for all $h \in S$, it follows that the number α is finite. Consider a point h in the set $\bar{S} := \{h \in C(x_0),\ \|h\| = 1\}$. In order to simplify the arguments suppose for the moment that problem (3.24) has an optimal solution $\bar{z} = \bar{z}(h)$. Since $\phi(\bar{z})$ is nonnegative, we have then that

$$2\bar{\tau}\|DG(x_0)h - \bar{z}\|^2 \le 2\bar{\tau}\|DG(x_0)h - \bar{z}\|^2 + \phi(\bar{z}) \le \phi(DG(x_0)h) \le \alpha,$$

and hence $\|DG(x_0)h - \bar{z}\|$ tends to zero as $\bar{\tau} \to +\infty$ uniformly in $h \in \bar{S}$. It follows then by lower semicontinuity of $\phi(\cdot)$, on $\bar{S}$, that for $\bar{\tau}$ large enough there is a positive constant c such that

$$\vartheta_{\bar{\tau}}(h) \ge \phi(\bar{z}(h)) \ge c, \quad \forall\, h \in \bar{S}.$$

Since for any h,

$$\vartheta_{\bar{\tau}}(h) \ge 2\bar{\tau}\|DG(x_0)h - \bar{z}\|^2,$$

by compactness arguments, condition (3.29) then holds for some $\bar{\tau}$ large enough. This completes the proof. ∎

Let us make the following remarks. Condition (iii) in the above theorem holds, for example, if the optimal solution x_0 is unique, Robinson's constraint qualification, at the point x_0, is satisfied and the so-called inf-compactness condition for the problem (P) holds (see, e.g., [1, pp. 263-264]).

If the set K is *polyhedral*, then it is second order regular and the sigma term in (3.24) vanishes. In that case, for $\bar{\tau}$ large enough, condition (3.29) is equivalent to the condition

$$D^2_{xx} L(x_0, \bar{\lambda})(h, h) > 0, \quad \forall\, h \in C(x_0) \setminus \{0\}. \tag{3.34}$$

If the set K is *cone reducible* (see [1, section 3.4.4] for a discussion of the concept of cone reducibility), then K is second order regular and the function (sigma term) $\phi(\cdot)$ is quadratic, and hence continuous, on $\mathcal{C}(x_0)$. A nontrivial example of a cone reducible set is (for any $p \in \mathbb{N}$) the cone $K := S^p_+$ of $p \times p$ positive semidefinite symmetric matrices. For $K := S^p_+$ the sigma term is quadratic and can be written explicitly.

4 Stability of augmented solutions

In this section we discuss stability of minimizers of the augmented Lagrangian under small perturbations of the corresponding augmented Lagrange multipliers. We assume in this section that the spaces $\mathcal{X}$ and $\mathcal{Y}$ are *finite* dimensional, the function $f(x)$ is *lower semicontinuous* and the mapping $G(x)$ is *continuous*.

Suppose that the problem (P) has nonempty set S^* of optimal solutions. Let $\bar{\lambda}$ be an augmented Lagrange multiplier, i.e., $\bar{\lambda} \in \partial v_{\bar{\tau}}(0)$ for some $\bar{\tau} \geq 0$. It follows then that $S^* \subset \arg\min_{x \in \mathcal{X}} \mathcal{L}(x, \bar{\lambda}, \bar{\tau})$. For a given $\lambda \in \mathcal{Y}$, consider the problem:

$$\operatorname*{Min}_{x \in \mathcal{X}} \mathcal{L}(x, \lambda, \bar{\tau}). \tag{4.1}$$

We study now the question what happens with the set of optimal solutions $S(\lambda) := \arg\min_{x \in \mathcal{X}} \mathcal{L}(x, \lambda, \bar{\tau})$ of the above problem under small perturbations of λ in a neighborhood of $\bar{\lambda}$.

It is said that the *inf-compactness* condition holds, for the problem (4.1), if there exist $\alpha \in \mathbb{R}$ and a bounded set $C \subset \mathcal{X}$ such that for every λ in a neighborhood of $\bar{\lambda}$ the level set $\{x \in \mathcal{X} : \mathcal{L}(x, \lambda, \bar{\tau}) \leq \alpha\}$ is nonempty and contained in C. The inf-compactness condition holds, for example, if the domain of f is bounded.

By the standard theory of sensitivity analysis we have that if the inf-compactness condition holds, then for all λ in a neighborhood of $\bar{\lambda}$ the set $S(\lambda)$ is nonempty and

$$\lim_{\lambda \to \bar{\lambda}} \left[\sup_{x \in S(\lambda)} \operatorname{dist}(x, S(\bar{\lambda})) \right] = 0. \tag{4.2}$$

Suppose that $S^* = \{x_0\}$, i.e., problem (P) has unique optimal solution x_0. We have then that $x_0 \in \arg\min_{x \in \mathcal{X}} \mathcal{L}(x, \bar{\lambda}, \bar{\tau})$. Suppose, further, that the quadratic growth condition (3.18) holds. It follows that if $\hat{x}(\lambda) \in S(\lambda) \cap \mathcal{N}$, then

$$\|\hat{x}(\lambda) - x_0\| \leq c^{-1} \kappa, \tag{4.3}$$

where $\kappa = \kappa(\lambda)$ is the Lipschitz constant, on the set $\mathcal{N}$, of the function

$$\psi_\lambda(\cdot) := \mathcal{L}(\cdot, \lambda, \bar{\tau}) - \mathcal{L}(\cdot, \bar{\lambda}, \bar{\tau}),$$

[1, Proposition 4.32]. Suppose now that the mapping $G : \mathcal{X} \to \mathcal{Y}$ is continuously differentiable. We have then by (3.6) and the chain rule of differentiation that the function $\psi_\lambda(\cdot)$ is differentiable and

$$\nabla \psi_\lambda(x) = 2\bar{\tau} \nabla G(x) \left[(2\bar{\tau})^{-1}(\lambda - \bar{\lambda}) - P_K(G(x) + (2\bar{\tau})^{-1}\lambda) + P_K(G(x) + (2\bar{\tau})^{-1}\bar{\lambda}) \right].$$

It follows that if $G(\cdot)$ is continuously differentiable, then the Lipschitz constant of $\psi_\lambda(\cdot)$, on a neighborhood of x_0, is of order $O(\|\lambda - \bar{\lambda}\|)$. We obtain the following result.

Theorem 4.1 *Suppose that the inf-compactness condition holds. Then for all λ in a neighborhood of $\bar{\lambda}$, the set $S(\lambda)$ is nonempty and* (4.2) *holds. Suppose, further, that $S^* = \{x_0\}$, $G(\cdot)$ is continuously differentiable, $\bar{\lambda} \in \partial v_{\bar{\tau}}(0)$ for some $\bar{\tau} \geq 0$, and the quadratic growth condition* (3.18) *is satisfied. Then the following holds*

$$\sup_{x \in S(\lambda) \cap \mathcal{N}} \|x - x_0\| = O(\|\lambda - \bar{\lambda}\|). \tag{4.4}$$

If, in addition to the assumptions of the above theorem, we assume that x_0 is the unique minimizer of $\mathcal{L}(\cdot, \bar{\lambda}, \bar{\tau})$, then the neighborhood $\mathcal{N}$ in formula (4.4) can be removed.

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