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SECOND-ORDER SUFFICIENT CONDITIONS FOR ERROR BOUNDS IN BANACH SPACES*

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Abstract. Recently, Huang and Ng presented second-order sufficient conditions for error bounds of continuous and Gâteaux differentiable functions in Banach spaces. Wu and Ye dropped the assumption of Huang and Ng on Gâteaux differentiability but required the space to be a Hilbert space. We carry on this research in two directions. First we extend Wu and Ye's result to some non-Hilbert spaces; second, same as Huang and Ng, we work on Banach spaces but provide different second-order sufficient conditions that may allow the function to be non-Gâteaux differentiable.

Key words. error bound, Hölder smooth subdifferential, proximal subdifferential, nonsmooth analysis

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1. Introduction. We consider error bounds for lower semicontinuous functions in Banach spaces. Let f be a proper lower semicontinuous function on a Banach space X. Our goal is to study conditions that guarantee the existence of positive constants γ and m such that

$$\text{dist}^m(x, S) \leq \gamma f(x)_+ \quad \text{for all } x \in X, \tag{1.1}$$

where $S := f^{-1}(-\infty, 0]$ and $f(x)_+ := \max\{f(x), 0\}$. We call (1.1) an error bound of order m. If (1.1) holds for $m = 1$, then the error bound is of Lipschitz type, which has been much discussed in the literature; see [7, 10, 11, 12, 13, 14, 15] and the book [5]. If the function f is convex, then there exist many equivalent characterizations for error bounds in terms of the first-order directional derivative or first-order subdifferential of function f. However, if the function is not convex, one usually gives only sufficient conditions in terms of various first-order generalized subdifferentials or first-order generalized directional derivatives [7, 8, 12, 14].

The first-order conditions used in the nonconvex case require that the generalized subdifferentials of f for all $x \notin S$ are bounded away from zero. Specifically, let ∂ be a certain generalized subdifferential of f and let

$$P(\alpha) := \{x \in X : x \notin S, \partial f(x) \cap B(0, \alpha) \neq \emptyset\},$$

where $B(x, \alpha)$ denotes a closed ball centered at x with radius α. In order to establish error bounds for nonconvex functions, it is usually assumed that $P(\alpha)$ is empty for some $\alpha > 0$; in other words, there exists a positive scalar α such that $\|\xi\| \geq \alpha$ for

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all $\xi \in \partial f(X \setminus S)$. This assumption is quite restrictive. One naturally asks whether there are certain conditions for error bound to hold, provided that

$$P(\alpha) \neq \emptyset \quad \text{for every } \alpha > 0. \tag{1.2}$$

If f is sufficiently smooth such that $\partial f(x)$ is a singleton and equals the derivative $f'(x)$ of f for every $x \notin S$, then (1.2) is equivalent to the existence of a sequence $\{x_n\}$ in $X \setminus S$ satisfying that $\lim_{n\to\infty} f'(x_n) = 0$.

Recently, some researchers have considered second-order sufficient conditions for error bounds of lower semicontinuous functions. Huang and Ng [7] proved that if f is Gâteaux differentiable and continuous in a Banach space, then an error bound of Lipschitz type holds under an assumption on certain second-order directional derivatives. Wu and Ye [15] removed this assumption and established a similar result. However, their result requires the space to be a Hilbert space. In this paper we present results that extend Wu and Ye's result to non-Hilbert spaces and results that extend Huang and Ng's work to possibly non-Gâteaux differentiable functions in Banach spaces.

2. Smoothness and subdifferentials. Let X be a Banach space. $B(x, r)$ and $B_r(x)$ denote the closed and the open ball centered at x with radius $r > 0$, respectively.

DEFINITION 2.1 (see [9]). *The modulus of smoothness $\rho_X(\tau)$, $\tau > 0$, of X is defined as*

$$\rho_X(\tau) := \sup\{(\|x+y\| + \|x-y\|)/2 - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\}.$$

X is said to be uniformly smooth if $\lim_{\tau\to 0+} \rho_X(\tau)/\tau = 0$. A uniformly smooth Banach space is said to have modulus of smoothness of power p *if for some $s > 0$,*

$$\rho_X(\tau) \leq s\tau^p \quad \textit{for all } \tau \geq 0. \tag{2.1}$$

Consider the example of $X = L_p$ ($p > 1$). For $\tau \geq 0$,

$$\rho_{L_p}(\tau) \leq \begin{cases} \tau^p/p, & p \in (1,2), \\ (p-1)\tau^2/2, & p \in [2,\infty). \end{cases}$$

Thus, L_p is uniformly smooth for $p > 1$ and has modulus of smoothness of power p for $p \in (1, 2)$ and of power 2 for $p \geq 2$. Let

$$J_p(x) := \{\xi \in X^* : \langle \xi, x\rangle = \|\xi\|\,\|x\|\,, \|\xi\| = \|x\|^{p-1}\}.$$

It is known that every uniformly smooth Banach space is reflexive, and if X is a reflexive Banach space, then $J_p(x)$ is the subdifferential of the convex function $x \mapsto \|x\|^p/p$. That is, $\xi \in J_p(x)$ if and only if

$$\|y\|^p/p - \|x\|^p/p \geq \langle \xi, y - x\rangle \quad \text{for all } y \in X.$$

In general, $J_p(x)$ is not necessarily a singleton; however, X is uniformly smooth if and only if $J_p(x)$ is single valued and uniformly continuous on bounded sets [4].

LEMMA 2.2. *Let X be a uniformly smooth Banach space, $x, y \in X$, and $m > 1$. Then*

$$\|y\|^m - \|x\|^m \geq m\,\langle J_m(x), y - x\rangle\,.$$

Proof. This is obvious from the definition of subdifferential inequality of convex functions. ∎

LEMMA 2.3. *Let X be a uniformly smooth Banach space and $x, y \in X$. If X has modulus of smoothness of power m for some $m > 1$, then there exists a constant $L > 0$ such that*

$$\langle J_m(x) - J_m(y), x - y\rangle \leq L\|x - y\|^m \quad \text{for all } x, y \in X. \tag{2.2}$$

Proof. See Theorem 2 and Remarks 4 and 5 in [16]. □

Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function with

$$\operatorname{dom} f := \{x \in X : f(x) < \infty\} \neq \emptyset.$$

Let us recall several well-known subdifferentials. Let $x \in \operatorname{dom} f$.

- The *Hölder-smooth subdifferential* of order $p > 1$ of f at x is defined as (see [2])

$$\partial_p^{HS} f(x) := \left\{\xi \in X^* : \liminf_{\|v\|\to 0} \frac{f(x+v) - f(x) - \langle \xi, v\rangle}{\|v\|^p} > -\infty\right\}.$$

  When $p = 2$, $\partial_p^{HS} f(x)$ is just the *Lipschitz-smooth subdifferential* $\partial^{LS} f(x)$ of f at x [1]:

$$\partial^{LS} f(x) := \left\{\xi \in X^* : \liminf_{\|v\|\to 0} \frac{f(x+v) - f(x) - \langle \xi, v\rangle}{\|v\|^2} > -\infty\right\}. \tag{2.3}$$

  When X is a Hilbert space and $p = 2$, $\partial_p^{HS} f(x)$ coincides with the proximal subdifferential $\partial^P f(x)$ [3]. Note that $\xi \in \partial^P f(x)$ if and only if there exist $\eta > 0$ and $\sigma > 0$ such that

$$f(x+v) - f(x) \geq \langle \xi, v\rangle - \sigma\|v\|^2 \quad \text{for all } v \in B(0, \eta).$$

- The *Fréchet subdifferential* of f at x is the set

$$\partial^F f(x) := \left\{\xi \in X^* : \liminf_{\|v\|\to 0} \frac{f(x+v) - f(x) - \langle \xi, v\rangle}{\|v\|} \geq 0\right\}.$$

- The *Clarke–Rockafellar subdifferential* of f at x is the set

$$\partial^{CR} f(x) := \left\{\xi \in X^* : \langle \xi, v\rangle \leq \sup_{\varepsilon>0} \limsup_{\substack{y \to_f x \\ t\downarrow 0}} \inf_{u \in B_\varepsilon(v)} \frac{f(y+tu) - f(y)}{t},\ \forall v \in X\right\},$$

  where $y \to^f x$ means $y \to x$ and $f(y) \to f(x)$; when f is locally Lipschitz at x, the Clarke–Rockafellar subdifferential coincides with the *Clarke subdifferential*

$$\partial^C f(x) := \left\{\xi \in X^* : \langle \xi, v\rangle \leq \limsup_{(y,t)\to(x,0^+)} \frac{f(y+tv) - f(y)}{t}, \forall v \in X\right\}.$$

- The *Hadamard subdifferential* of f at x is the set

$$\partial^H f(x) := \left\{\xi \in X^* : \langle \xi, v\rangle \leq \liminf_{(u,t)\to(v,0^+)} \frac{f(x+tu) - f(x)}{t},\ \forall v \in X\right\}.$$

  When f is locally Lipschitz at x, the Hadamard subdifferential coincides with the *Gâteaux subdifferential*

$$\partial^G f(x) := \left\{\xi \in X^* : \langle \xi, v\rangle \leq \liminf_{t\to 0^+} \frac{f(x+tv) - f(x)}{t}, \forall v \in X\right\}.$$

It is straightforward to verify that for $p > 1$,

$$\partial_p^{HS} f(x) \subset \partial^F f(x) \subset \partial^H f(x) \subset \partial^{CR} f(x). \tag{2.4}$$

PROPOSITION 2.4. *Let g be a continuous function on a Banach space X. Suppose that $\partial_p^{HS} g(x)$ and $\partial_p^{HS}(-g)(x)$ are both nonempty. Then $\partial_p^{HS} g(x)$ is equal to $-\partial_p^{HS}(-g)(x)$ and $\partial_p^{HS} g(x)$ is a singleton.*

Proof. Let $\xi \in \partial_p^{HS} g(x)$ and $x^* \in \partial_p^{HS}(-g)(x)$. From the definition of the Hölder-smooth subdifferential, there exist $\sigma > 0$ and $\eta > 0$ such that for all $v \in B(0,\eta)$,

$$g(x+v) - g(x) \geq \langle \xi, v\rangle - (\sigma/2)\|v\|^p,$$
$$-g(x+v) + g(x) \geq \langle x^*, v\rangle - (\sigma/2)\|v\|^p.$$

Adding these two expressions together, we have

$$\langle \xi + x^*, v\rangle \leq \sigma \|v\|^p \quad \text{for all } v \in B(0,\eta),$$

which implies that $\xi + x^* = 0$ as $p > 1$. Since $\xi \in \partial_p^{HS} g(x)$ and $x^* \in \partial_p^{HS}(-g)(x)$ are arbitrary, $\partial_p^{HS} g(x)$ is equal to $-\partial_p^{HS}(-g)(x)$ and is a singleton. ∎

PROPOSITION 2.5. *The subdifferential ∂_p^{HS} has the following properties:*

(P1) *$\partial_p^{HS} f(x)$ coincides with the subdifferential in the sense of convex analysis whenever f is convex;*

(P2) *$0 \in \partial_p^{HS} f(x)$ whenever $x \in \operatorname{dom} f$ is a local minimum of f;*

(P3) *$\partial_p^{HS}(f+g)(x) \subset \partial_p^{HS} f(x) + \partial_p^{HS} g(x)$ whenever g is a continuous function with the property that $\partial_p^{HS} g(x)$ and $\partial_p^{HS}(-g)(x)$ are both nonempty.*

Proof. (P1) Let g be a convex function and $x \in \operatorname{dom} g$. Just observe that for a convex function the Clarke–Rockafellar subdifferential and the usual (Fenchel) subdifferential in convex analysis coincide for lower semicontinuous functions and that the Fenchel subdifferential is obviously contained in $\partial_p^{HS} g(x)$. The conclusion follows immediately from (2.4).

(P2) It is obvious from the definition of ∂_p^{HS}.

(P3) Note that

$$\partial_p^{HS} f(x) = \partial_p^{HS}(f+g-g)(x) \supset \partial_p^{HS}(f+g)(x) + \partial_p^{HS}(-g)(x), \tag{2.5}$$

where the inclusion relation is from the definition of the Hölder-smooth subdifferential. Since g is continuous and $\partial_p^{HS} g(x)$ and $\partial_p^{HS}(-g)(x)$ are both nonempty, by virtue of Proposition 2.4, $\partial_p^{HS}(-g)(x)$ is a singleton and $\partial_p^{HS}(-g)(x) = -\partial_p^{HS} g(x)$. This together with (2.5) yield the conclusion. ∎

PROPOSITION 2.6. *If X is a uniformly smooth Banach space which has modulus of smoothness of power p for some $p > 1$ and $x \neq 0$, then the Hölder-smooth subdifferential of order p of the functions $\|x\|^p/p$ and $-\|x\|^p/p$ are nonempty and $\partial_p^{HS}(-\|\cdot\|^p/p)(x) = -J_p(x)$.*

Proof. Since X is uniformly smooth, the function $\|\cdot\|$ and hence the convex function $\|\cdot\|^p/p$ are Fréchet differentiable at x. Therefore $\partial_p^{HS}(\|\cdot\|)(x)$ is nonempty by Proposition 2.5. Now we prove that $\partial_p^{HS}(-\|\cdot\|^p/p)(x)$ is nonempty. Since $J_p(x)$ is the subdifferential of $\|x\|^p/p$ in the sense of convex analysis, for $v \neq 0$,

$$\frac{-\|x+v\|^p/p + \|x\|^p/p - \langle -J_p(x), v\rangle}{\|v\|^p} \geq \frac{\langle J_p(x) - J_p(x+v), v\rangle}{\|v\|^p} \geq -L,$$

where the last inequality follows from Lemma 2.3 and L is the constant that appeared in Lemma 2.3. This proves that $-J_p(x)$ belongs to $\partial_p^{HS}(-\|\cdot\|^p/p)(x)$ by the definition of the Hölder-smooth subdifferential. ∎

3. Error bounds in smooth Banach spaces. The following result generalizes the second-order sufficient condition for error bounds established in [15] from the Hilbert space to smooth Banach spaces.

THEOREM 3.1. *Let X be a uniformly smooth Banach space which has modulus of smoothness of power m for some $m > 1$, and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function. Suppose that there exists $\delta > 0$ such that for all $x \in f^{-1}(0, \infty)$,*

$$\liminf_{\|u\|\to 1, t\downarrow 0} \frac{f(x+tu) - f(x) - t\langle \xi, u\rangle}{t^m} < -\delta \quad \text{for each } \xi \in \partial_m^{HS} f(x). \tag{3.1}$$

Then

$$\operatorname{dist}^m(x, S) \le (mL/\delta)\, f(x)_+ \quad \text{for all } x \in X, \tag{3.2}$$

where L is the constant that appeared in (2.2).

Proof. Write γ for mL/δ. Suppose that the conclusion does not hold: there exists some u with $f(u) > 0$ such that

$$\operatorname{dist}^m(u, S) > \gamma\, f(u).$$

Then we can find $t > 1$ such that $\operatorname{dist}^m(u, S) > t\gamma\, f(u)$, and hence

$$f(u) = f(u)_+ < \inf_{x\in X} f(x)_+ + \gamma^{-1} c, \tag{3.3}$$

where $c := t\gamma\, f(u)$. Applying the Borwein–Preiss smooth variational principle [2], we obtain the existence of some $v \in X$ such that

$$\|u - v\| < \sqrt[m]{c} \quad \text{and} \tag{3.4}$$

$$f(v)_+ + \gamma^{-1}\Delta_m(v) \le f(x)_+ + \gamma^{-1}\Delta_m(x) \quad \text{for all } x \in X, \tag{3.5}$$

where $\Delta_m(x) := \sum_{k=1}^{\infty} \mu_k \|x - v_k\|^m$ for some sequence $\{v_k\}$ converging to v and some sequence $\{\mu_k\}$ satisfying $\mu_k > 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

It follows from (3.4) and the choice of u that $v \notin S$. Hence v is a global minimizer of the function $f(x) + \gamma^{-1}\Delta_m(x)$ and hence a global minimizer of the function $\gamma m^{-1} f(x) + m^{-1}\Delta_m(x)$ over the open set $X \setminus S$. In view of the definition of Hölder-smooth subdifferential ∂_m^{HS}, it follows that

$$0 \in \partial_m^{HS}(\gamma m^{-1} f + m^{-1}\Delta_m)(v). \tag{3.6}$$

Clearly $m^{-1}\Delta_m(x)$ is a real valued continuous convex function. Hence $\partial_m^{HS}(m^{-1}\Delta_m)(v)$ coincides with the subdifferential in the sense of convex analysis by Proposition 2.5 and so is nonempty. Since the space X is uniformly smooth, it follows that for every x, $J_m(x - v_k)$ is a singleton for each k and the sequence $\{J_m(x - v_k)\}_{k=1}^{\infty}$ is bounded. Thus, $m^{-1}\Delta_m(x)$ is Fréchet differentiable with its Fréchet derivative $(m^{-1}\Delta_m)'(x) = \sum_{k=1}^{\infty} \mu_k J_m(x - v_k)$. Since $\partial_m^{HS}(m^{-1}\Delta_m)(v)$ is nonempty, it follows that

$$\partial_m^{HS}(m^{-1}\Delta_m)(v) = \left\{(m^{-1}\Delta_m)'(v)\right\}. \tag{3.7}$$

We claim that $\partial_m^{HS}(-m^{-1}\Delta_m)(v)$ contains $-(m^{-1}\Delta_m)'(v)$ and hence is nonempty. This together with (3.6), Propositions 2.4 and 2.5, and (3.7) yields that

$$\xi := -m\gamma^{-1} \sum_{k=1}^{\infty} \mu_k J_m(v - v_k) \in \partial_m^{HS} f(v). \tag{3.8}$$

Indeed,

$$\begin{aligned}
&\liminf_{h\to 0} \frac{(-m^{-1}\Delta_m)(v+h) - (-m^{-1}\Delta_m)(v) - \left\langle (-m^{-1}\Delta_m)'(v), h\right\rangle}{\|h\|^m} \\
&= \liminf_{h\to 0} \frac{\left\langle (-m^{-1}\Delta_m)'(v+\theta(h)h), h\right\rangle - \left\langle (-m^{-1}\Delta_m)'(v), h\right\rangle}{\|h\|^m} \quad (0<\theta(h)<1) \\
&= \liminf_{h\to 0} \frac{\sum_{k=1}^{\infty} \mu_k \left\langle J_m(v-v_k) - J_m(v+\theta(h)h - v_k), h\right\rangle}{\|h\|^m} \\
&\geq \liminf_{h\to 0} -L\theta(h)^{m-1} \geq -L > -\infty,
\end{aligned}$$

where the first equality is from the mean value theorem and the first inequality follows from Lemma 2.3 and the facts of $\mu_k > 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$. In view of the definition of Hölder-smooth subdifferential ∂_m^{HS}, it follows that $-(m^{-1}\Delta_m)'(v) \in \partial_m^{HS}(-m^{-1}\Delta_m)(v)$.

By (3.8) and the assumption (3.1), there exist sequences $t_n \to 0+$ and $\|u_n\| \to 1$ such that

$$\lim_{n\to\infty} \frac{f(v+t_n u_n) - f(v) - t_n \langle \xi, u_n\rangle}{t_n^m} < -\delta = -mL\gamma^{-1}. \tag{3.9}$$

Since $X \setminus S$ is an open set as f is lower semicontinuous, we have $f(v+t_n u_n) > 0$ for sufficiently large n. It follows from (3.5) that

$$\begin{aligned}
&\frac{f(v+t_n u_n) - f(v) - t_n \langle \xi, u_n\rangle}{t_n^m} \\
&= \frac{f(v+t_n u_n) - f(v) + m\gamma^{-1} t_n \sum_{k=1}^{\infty} \mu_k \langle J_m(v-v_k), u_n\rangle}{t_n^m} \\
&\geq \frac{\sum_{k=1}^{\infty} \mu_k \left\{ \|v-v_k\|^m - \|v+t_n u_n - v_k\|^m \right\} + m \sum_{k=1}^{\infty} \mu_k \langle J_m(v-v_k), t_n u_n\rangle}{\gamma t_n^m} \\
&\geq m\gamma^{-1} t_n^{-m} \sum_{k=1}^{\infty} \mu_k \langle J_m(v-v_k) - J_m(v+t_n u_n - v_k), t_n u_n\rangle \\
&\geq -mL\gamma^{-1} \|u_n\|^m \to -mL\gamma^{-1} = -\delta \ (\text{ as } n\to\infty),
\end{aligned}$$

where the second inequality follows from Lemma 2.2 and the third inequality follows from Lemma 2.3. This contradicts (3.9). ∎

In view of the assumption (3.1), it is straightforward to see that if the $\partial_p^{HS} f(x)$ is replaced by a larger set such as $\partial^F f(x)$, $\partial^H f(x)$, or $\partial^{CR} f(x)$ (see (2.4)), then the condition becomes more stringent. In other words, our requirement on the subdifferential is fairly weak.

COROLLARY 3.2. *Let X be a uniformly smooth Banach space which has modulus of smoothness of power 2, and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function. If there exists $\delta > 0$ such that for all $x \in f^{-1}(0,\infty)$ and all $\xi \in \partial^{LS} f(x)$,*

$$\liminf_{\|u\|\to 1, t\downarrow 0} \frac{f(x+tu) - f(x) - t\langle \xi, u\rangle}{t^2} < -\delta,$$

then

$$\operatorname{dist}^2(x,S) \leq (2L/\delta)\, f(x)_+ \quad \textit{for all } x \in X.$$

Proof. Since $p = 2$, the Hölder subdifferential ∂_p^{HS} coincides with the Lipschitz-smooth subdifferential ∂^{LS}. The conclusion thus follows immediately from Theorem 3.1. □

Remark 3.1. Since all Hilbert spaces are uniformly smooth with modulus of smoothness of power 2 (see [9]) and since when X is a Hilbert space $\partial^{LS} f(x)$ coincides with the proximal subdifferential $\partial^P f(x)$, Corollary 3.2 generalizes Theorem 3.1 in [15] for its $\varepsilon = \infty$. Moreover, there exist Banach spaces, say $L^p(\mu)$ for $p \geq 2$, which are uniformly smooth with modulus of smoothness of power 2 but are not Hilbert spaces [9]. Therefore Corollary 3.2 is applicable to a broader class of spaces than [15, Theorem 3.1]. The same as what was done in [15], our results can also be verified for general $\varepsilon > 0$. We omit the details for brevity.

From the argument of Theorem 3.1, it can be seen that one can replace the Hölder smooth subdifferential ∂_m^{HS} of f by some other classes of subdifferentials. Let us define an abstract subdifferential in the following.

DEFINITION 3.3 (see [1]). *An abstract subdifferential, denoted by ∂, is any operator that associates a subset $\partial f(x) \subset X^*$ to a lower semicontinuous function $f : X \to \mathbb{R} \cup \{\infty\}$ and a point $x \in X$, satisfying the following properties:*

(P1) *$\partial f(x)$ coincides with the subdifferential in the sense of convex analysis whenever f is convex;*

(P2) *$0 \in \partial f(x)$ whenever $x \in \operatorname{dom} f$ is a local minimum of f;*

(P3) *$\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$ whenever g is a real valued convex continuous function which satisfies $\partial g(x)$ and $\partial(-g)(x)$ are both nonempty.*

Paper [1] provides various classes of subdifferentials satisfying the above properties (P1)–(P3)—for example, the Hadamard subdifferential, the Gâteaux subdifferential, the Fréchet subdifferential, and the Clarke–Rockafellar subdifferential.

For $p > 1$, we denote by Γ_p all the functions of the form

$$\Gamma(x) := \frac{1}{p} \sum_{k=1}^{\infty} \mu_k \left\| x - u_k \right\|^p \quad \text{for all } x \in X, \tag{3.10}$$

where $\{u_k\}$ is any convergent sequence in X and $\{\mu_k\}$ is any sequence of nonnegative scalars satisfying $\sum_{k=1}^{\infty} \mu_k = 1$. Clearly, each function in Γ_p is a real valued continuous convex function.

THEOREM 3.4. *Let X be a uniformly smooth Banach space which has modulus of smoothness of power $m > 1$ and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function. Let ∂ be an abstract subdifferential satisfying properties* (P1)–(P3) *in Definition* 3.3 *and an additional property:*

(P4) *$\partial(-\Gamma)(x)$ is nonempty for each $\Gamma \in \Gamma_m$.*

If there exists $\delta > 0$ such that for all $x \in f^{-1}(0, \infty)$ and all $\xi \in \partial f(x)$,

$$\liminf_{\|u\| \to 1, t \downarrow 0} \frac{f(x + tu) - f(x) - t \langle \xi, u \rangle}{t^m} < -\delta,$$

then

$$\operatorname{dist}^m(x, S) \leq (mL/\delta)\, f(x)_+ \quad \textit{for all } x \in X.$$

Proof. After checking the proof of Theorem 3.1, we know the key role played by the subdifferential is the part from (3.6) to (3.8). Since each $\Gamma(x)$ is a continuous real valued convex function, $\partial\Gamma(x)$ is nonempty. In view of the property (P4) and (P3), one can establish (3.8) in a similar way. The remaining proof is similar to the proof of Theorem 3.1. ◻

The above theorems establish m-order error bounds for lower semicontinuous functions in certain classes of Banach spaces. As a corollary of Theorem 3.1, we give an error bound of order one whose proof is similar to that of Theorem 3.3 in [15]. Recall that S is the set $f^{-1}(-\infty, 0]$, and define

$$P(\alpha) := \{x \in X \setminus S : \partial_m^{HS} f(x) \cap B(0, \alpha) \neq \emptyset\} \quad \text{for } \alpha > 0. \tag{3.11}$$

THEOREM 3.5. *Let X be a uniformly smooth Banach space which has modulus of smoothness of power m for some $m > 1$ and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function. Suppose that the following two conditions hold.*

(i) *$P(\alpha) \subset f^{-1}(\beta, \infty)$ for some $\alpha > 0$ and some $\beta > 0$.*

(ii) *There exists $\delta > 0$ such that for all $x \in f^{-1}(\beta, \infty)$ and all $\xi \in \partial_m^{HS} f(x)$,*

$$\liminf_{\|u\| \to 1, t \downarrow 0} \frac{f(x + tu) - f(x) - t \langle \xi, u \rangle}{t^m} < -\delta.$$

Then there exists $c > 0$ such that

$$\operatorname{dist}(x, S) \leq c\, f(x)_+ \quad \textit{for all } x \in X.$$

4. Error bounds in general Banach spaces. In the last section, we have established second-order sufficient conditions for error bounds of lower semicontinuous functions in smooth Banach spaces. In what follows we will provide different second-order sufficient conditions for error bounds in general Banach spaces. The result of this section generalizes that in [7], which gives second-order sufficient conditions for error bounds in general Banach spaces but requires the function to be Gâteaux differentiable. Our results show that the assumption of Gâteaux differentiability can be removed. Before that, we need to define second-order directional derivative. Let X be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function. For $x, u, v \in X$, we define respectively Hadamard directional derivative and a second-order directional derivative:

$$f'_-(x; u) := \liminf_{v \to u,\, t \downarrow 0} \frac{f(x + tv) - f(x)}{t};$$

$$d^2_- f(x; u, v) := \liminf_{t \to 0+} \frac{f(x + tu + t^2 v) - f(x) - t f'(x)u}{t^2}$$

whenever f is Gâteaux differentiable.

It can be seen that if f is Gâteaux differentiable at x with $f'(x)$ being the Gâteaux derivative, then $f'_-(x; u) \leq f'(x)u$ for every u; the equality holds if in addition f is locally Lipschitz at x. If f is twice continuously differentiable, then

$$d^2_- f(x; u, 0) = (1/2) \nabla^2 f(x)(u, u),$$

where $\nabla^2 f(x)$ denotes the second-order derivative of f at x.

For $\varepsilon > 0$, we define a set

$$D(\varepsilon) := \left\{x \in X : x \notin S \text{ and } \inf_{\|u\|=1} f'_-(x; u) \geq -\varepsilon\right\}.$$

If f is Gâteaux differentiable on X, then

$$D(\varepsilon) \subset \{x \in X : x \notin S \text{ and } \|f'(x)\| \leq \varepsilon\} =: \mathfrak{D}(\varepsilon),$$

where the set $\mathfrak{D}(\varepsilon)$ is introduced and used in [7] for studying second-order sufficient conditions for continuous and Gâteaux differentiable functions to have error bounds.

The following lemma [12, Lemma 2.3] is a straightforward consequence of Theorem 2(ii) in [6].

LEMMA 4.1. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function. If there exists $\gamma > 0$ such that for every $x \in f^{-1}(0, \infty)$ there is $y \in f^{-1}[0, \infty)$ such that*

$$f(x) - f(y) \geq \gamma \|x - y\| > 0,$$

then $\operatorname{dist}(x, S) \leq \gamma^{-1} f(x)_+$ *for all $x \in X$.*

THEOREM 4.2. *Let $f : X \to \mathbb{R}$ be a continuous function. Suppose that there exist positive scalars r, ρ, and δ such that the following conditions hold:*

(i) *$D(\rho) \subset f^{-1}(r, \infty)$;*

(ii) $\limsup_{t\to 0+} \sup_{x\in D(\rho)} \inf_{\|u\|=1} \frac{f(x+tu)-f(x)+tf'_-(x;-u)}{t^2} < -\delta$.

Then there exists $\gamma > 0$ such that $\operatorname{dist}(x, S) \leq \gamma^{-1} f(x)_+$ *for all $x \in X$.*

Proof. We need to consider only those points x not in S. In view of the assumption (ii), there exists $\beta \in (0, 1/2]$ such that for every $t \in (0, \beta)$ and every $x \in D(\rho)$, a unit vector u (dependent on t and x) exists and satisfies that

$$\frac{f(x + tu) - f(x) + tf'_-(x; -u)}{t^2} < -\delta. \tag{4.1}$$

Take $\varepsilon = \min\{\rho, \beta\delta/4\}$ and $\gamma = \min\{r, \varepsilon/2\}$.

Let $x \in D(\varepsilon)$ be such that $\operatorname{dist}(x, S) \geq 1$. Put $\lambda = \beta/2$. It follows that $x + \lambda u \notin S$ for any unit vector u. Since $\varepsilon \leq \rho$, $x \in D(\varepsilon) \subset D(\rho)$, it follows from (4.1) that there exists a unit vector u_λ such that

$$f(x + \lambda u_\lambda) - f(x) + \lambda f'_-(x; -u_\lambda) < -\lambda^2\delta.$$

In view of the definition of $D(\varepsilon)$, $x \in D(\varepsilon)$ implies that $f'_-(x; -u_\lambda) \geq -\varepsilon$. Therefore,

$$f(x) - f(x + \lambda u_\lambda) \geq \lambda^2\delta - \lambda\varepsilon \geq \gamma\lambda = \gamma \|x - (x + \lambda u_\lambda)\|.$$

For $x \in D(\varepsilon)$ and $\operatorname{dist}(x, S) < 1$, there exists $y \in S$ such that $\|x - y\| < 1$. Since f is continuous, y can be chosen to satisfy $f(y) = 0$. Since $x \in D(\varepsilon)$ and $D(\varepsilon) \subset f^{-1}(r, \infty)$, one has $f(x) > r$. It follows that

$$f(x) - f(y) \geq r > r \|x - y\| \geq \gamma \|x - y\| > 0.$$

For $x \notin D(\varepsilon)$, we have $f'_-(x; u) < -\varepsilon$ for some unit vector u. It follows that there exist a sequence of positive scalars $\{t_n\}$ converging to zero and a sequence $\{u_n\}$ converging to u such that for sufficiently large n,

$$f(x + t_n u_n) - f(x) < -\varepsilon t_n.$$

Since $\gamma < \varepsilon$ and $\|u_n\| \to 1$, $\gamma \|u_n\| < \varepsilon$ for large enough n. This implies that

$$f(x) - f(x + t_n u_n) > \varepsilon t_n \geq \gamma t_n \|u_n\| = \gamma \|x - (x + t_n u_n)\|$$

for sufficiently large n.

Thus, we have shown that for each $x \notin S$, there exists $y \in f^{-1}[0,\infty)$ such that $f(x) - f(y) \geq \gamma \|x - y\|$. Then, by applying Lemma 4.1, we obtain the desired conclusion. □

Huang and Ng [7] considered error bounds in general Banach spaces for a function which is Gâteaux differentiable and continuous. Besides the assumption (i) of Theorem 4.2, [7] requires another condition: There exist $\beta > 0$ and $\delta > 0$ such that for all $x \in \mathfrak{D}(\rho)$,

$$\inf_{\|u\|=1} \sup_{t\in[0,\beta)} d_-^2 f(x+tu; u, 0) < -\delta. \tag{4.2}$$

Because f is Gâteaux differentiable and continuous, $f'_-(x; -u) \leq -f'(x)u$. It follows from [7, Theorem 3.1] that the condition (4.2) implies the existence of $\beta > 0$ and $\delta > 0$ such that for all $x \in \mathfrak{D}(\rho)$,

$$\inf_{\|u\|=1} \sup_{t\in(0,\beta)} \frac{f(x+tu) - f(x) + tf'_-(x; -u)}{t^2} < -\delta. \tag{4.3}$$

Note that our assumption (ii) in Theorem 4.2 is that there exist $\beta > 0$ and $\delta > 0$ such that for all $x \in D(\rho)$,

$$\sup_{t\in(0,\beta)} \inf_{\|u\|=1} \frac{f(x+tu) - f(x) + tf'_-(x; -u)}{t^2} < -\delta. \tag{4.4}$$

Since $D(\rho) \subset \mathfrak{D}(\rho)$, it is straightforward that (4.3) and hence (4.2) imply (4.4). The latter is a restatement of our assumption (ii), which is therefore less restrictive than the assumption (4.2) as our assumption (ii) also allows f to be non-Gâteaux differentiable.

REFERENCES

[1] D. Aussel, J.-N. Corvellec, and M. Lassonde, *Mean value property and subdifferential criteria for lower semicontinuous functions*, Trans. Amer. Math. Soc., 347 (1995), pp. 4147–4161.

[2] J. M. Borwein and D. Preiss, *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc., 303 (1987), pp. 517–527.

[3] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York, 1998.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[6] A. Hamel, *Remarks to an equivalent formulation of Ekeland's variational principle*, Optimization, 31 (1994), pp. 233–238.

[7] L. R. Huang and K. F. Ng, *On first- and second-order conditions for error bounds*, SIAM J. Optim., 14 (2004), pp. 1057–1073.

[8] A. D. Ioffe, *Regular points of Lipschitz functions*, Trans. Amer. Math. Soc., 251 (1979), pp. 61–69.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vol. II, Ergeb. Math. Grenzgeb. 97, Springer-Verlag, Berlin, 1979.

[10] K. F. Ng and W. H. Yang, *Error bounds for abstract linear inequality systems*, SIAM J. Optim., 13 (2002), pp. 24–43.

[11] K. F. Ng and X. Y. Zheng, *Global error bounds with fractional exponents*, Math. Program., 88 (2000), pp. 357–370.

[12] K. F. Ng and X. Y. Zheng, *Error bounds for lower semicontinuous functions in normed spaces*, SIAM J. Optim., 12 (2001), pp. 1–17.

[13] Z. Wu and J. J. Ye, *Sufficient conditions for error bounds*, SIAM J. Optim., 12 (2001), pp. 421–435.

[14] Z. Wu and J. J. Ye, *On error bounds for lower semicontinuous functions*, Math. Program., 92 (2002), pp. 301–314.

[15] Z. Wu and J. J. Ye, *First-order and second-order conditions for error bounds*, SIAM J. Optim., 14 (2003), pp. 621–645.

[16] Z. B. Xu and G. F. Roach, *Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces*, J. Math. Anal. Appl., 157 (1991), pp. 189–210.