Optimal Logistics Outsourcing: A geometric Approach

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Abstract

In this paper, we introduce a very simple geometric approach to study logistics outsourcing issues in a principal-agent setting. We propose a method to determine the optimal outsourcing contract, using the notion of general cost sharing function. This simplifies the discussion on optimal contract design significantly. More importantly, for several simple environments, this restriction of the contract space does not lead to any deterioration in performance as it includes the optimal contract in the feasible contract options. The simplicity of this approach makes it more adaptable to analyzing logistics outsourcing contract options. We use a newsvendor problem to demonstrate the applicability of this approach. Some extension of this approach is proposed to complete a further insight of this approach.

Keywords: logistics outsourcing, geometric approach, optimal incentive contract, newsvendor

1 Introduction

Logistics outsourcing is a popular approach used by many companies to manage their supply chains, and has shown to have direct and significant impact on the revenue, especially for those firms suffering from high demand and component spot price volatility (e.g., firms in apparel or electronics industry). This is especially essential if the logistics functions are deemed to be non-core activities. These firms also benefit from professional logistics management skills and scale effect of the agents (or strategic partners) to whom the logistics functions are outsourced to.

One the most critical issue in logistics outsourcing is the development of the contractual terms, which dictates the payment terms and conditions to the agents, for the activities carried out on behalf of the customers (hereafter referred to as principals). Unfortunately, the issue of information asymmetry (the cost structure and effort levels of the agents may not be observable to the principals) has been a stumbling roadblock to many logistics outsourcing opportunities. For instance, it is not uncommon to find staff members of large institutions complaining of high travel cost charged by their travel agents, even though the contractual terms dictate that the agents must find the most economical and direct routes for the trips. The lack of efforts on the part of the agents to economize on the routes, if the payments terms are dictated strictly by asset utilization or miles traveled.

What is the most appropriate way to design an outsourcing contract, addressing the issue of information asymmetry? To answer this question, we borrow tools and concepts from the procurement literature. The development of optimal contract theory and incentive schemes have been a milestone in the procurement contracting area. Laffont and Tirole (1986)’s influential paper demonstrated the characteristics and existence of the optimal mechanism in public firm regulation. Subsequently in Laffont and Tirole (1993), they extended the model to cover the contracting issues between private firms. However, the optimal mechanism, involving a menu of contracts, is very complex and not suitable for practical implementation. A few recent papers (Rogerson 2003, Chu and Sappington 2007) focus on the performance of simple incentive contracts, and demonstrated a very important insight - that a combination of some simple procurement contracts, such as fix cost contract, cost reimbursement contract, linear cost-sharing contract and so on, can secure a large fraction of the surplus that a fully optimal contract can secure. Unfortunately, the exposition in this literature focused on very simple settings, and very few papers extended the analysis to consider supply chain issues with its associated complexities. This problem is exacerbated by the difficulty in solving for the optimal contract, especially in complex environments.

In this paper, we introduce a very simple geometric approach to study logistics outsourcing issues in a principal-
agent setting. We propose a method to determine the optimal outsourcing contract, using the notion of general cost sharing function. This simplifies the discussion on optimal contract design significantly. More importantly, for several simple environment, this restriction of the contract space does not lead to any deterioration in performance as it includes the optimal contract in the feasible contract options. The simplicity of this approach makes it more adaptable to analyzing logistics outsourcing contract options. We use a newsvendor problem to demonstrate the applicability of this approach.

2 Model and Approach

Consider an agent managing the logistics operations for a principal. The cost of the daily operation is denoted by \( n \). The cost \( n \) is affected by many factors in practice, due to variation in daily demand and operational characteristics. The agent possesses the information on the operational cost \( n \), but this information is not privy to the principal, who believes \( n \) to lie in \([\bar{n}, \overline{n}]\) with probability density function \( f(n) \) and cumulative distribution function \( F(n) \). With additional efforts, through better route planning or demand forecasting, it may be possible for the agent to reduce the daily operational cost to \( n - e \), at a disutility of \( \psi(e) \), where \( \psi(\cdot) \) is strictly increasing in \( e \). Note that the actual cost \( n - e \) is observable to the principal, but not the effort \( e \) and the original cost \( n \). For ease of exposition, we assume \( \overline{n} \geq (\psi')^{-1}(1) \).

How should the principal write into the contract to induce the optimal efforts from the agent that will lead to the smallest operational cost? It is clear that the commonly used approach of fixed cost contract does not work - it only attracts agents who can already operate at a cost lower than the fixed cost awarded, but does not encourage the agent to put in additional effort to bring down the daily operational cost. The uncertainty in the daily operational cost \( n \) will also impose additional risk on the part of the agent, and a risk premium must be written into the contract to protect the agent. Similarly, a full cost reimbursement (with built in profit margin), calculated using complex formulas on asset utilization and activity based costing approach, will not induce the optimal effort on the part of the agent.

Let \( \psi(e) \) denote the marginal cost of effort at level \( e \).

Thus

\[ \psi(e) = \int_0^e \psi'(t)dt. \]

For ease of exposition, we assume \( \psi'(t) = 0 \) for \( t < 0 \). i.e., there is no cost in inflating the operational cost. We define as variable the cost sharing function \( V(t) \), where \( V(t) \) denote the cost borne by the agent when the actual operational cost \( n - e = t \). Let \( V'(t) \in [0, 1] \) denote the marginal rate of cost sharing. Thus

\[ V(t) = \int_0^t V'(s)ds. \]

In a similar vein, let \( F \) denote the lump sum payment awarded to the agent for the outsourcing service. In this setting, when the operational cost is \( n \), the agent will have to determine the corresponding effort level to obtain actual operational cost of \( n - e \), so as to receive the payment \( F \) (reward for operating) and \( (n - e - V(n - e)) \) (share of the operational cost from the principal), to compensate for the disutility function for effort \( \psi(e) \), and the share of the cost \( V(n - e) \) for daily operational cost of \( n - e \). We call this the class of nonlinear cost sharing contracts.

Our challenge is to find non-decreasing functions \( \{V(t)\} \) and lump sum payment \( F \) such that

- Objective: the expected payment from the principal to the agent is minimum;
- Participation: for all realization of \( n \), there is an effort level \( e \) such that the payment received by the agent \( F + (n - e - V(n - e)) \) is always greater than the total cost incurred, \( (n - e) + \psi(e) \). i.e.,

\[ F - V(n - e) - \psi(e) \geq 0. \]

- Boundary condition: For all effort level \( e \),

\[ F - V(n - e) - \psi(e) \leq 0. \]

- Incentive Compatible: Given \( n \), the agent chooses an effort level \( e(n) \) to maximize

\[ F + (n - e - V(n - e)) - ((n - e) + \psi(e)) = F - V(n - e) - \psi(e). \]

- Truth Revealing: Given \( n \) and the corresponding effort level \( e(n) \), it must be the case that for all \( \hat{n} \neq n \),

\[ F - V(n - e(n)) - \psi(e(n)) \geq F - V(\hat{n} - e(\hat{n})) - \psi(e(\hat{n}|n)) \]

where

\[ n - e(\hat{n}|n) = \hat{n} - e(\hat{n}). \]

The participation and incentive compatibility constraints ensure that

\[ F \geq V(n - e(n)) + \psi(e(n)) \]

\[ = \int_0^{n - e(n)} V'(s)ds + \int_{n - e(n)}^{n} \psi'(n - s)ds. \]

Note that given \( n \), the term on the right hand side is minimize at effort level \( e(n) \) when

\[ V'(n - e(n)) = \psi'(e(n)). \]
$e(n)$ can be determined in a geometric manner, once $n - e(n)$ is known:

Furthermore, the right hand side is monotonically increasing in $n$, hence using the boundary condition,

$$F = V(\pi - e(\pi)) + \psi(e(\pi))$$

$$= \int_0^{\pi - e(\pi)} V'(s)ds + \int_{e(\pi)}^\pi \psi'(\pi - s)ds,$$

where

$$V'(\pi - e(\pi)) = \psi'(e(\pi)).$$

The optimal contract design problem for the principal can now be cast as

$$\min_{V'(t)} \left( F + \int_0^{n - e(n)} (1 - V'(t))dt \right)$$

$$= \min_{V'(t)} \left( \int_0^{\pi - e(\pi)} V'(s)ds \right. $$

$$\left. + \int_{e(\pi)}^\pi \psi'(\pi - s)ds + E_n \left( \int_0^{n - e(n)} (1 - V'(t))dt \right) \right),$$

where $e(n)$ satisfies

$$V'(n - e(n)) = \psi'(e(n)), \ \forall \ n \in [n, \pi].$$

Note that when $V'(t) = 0$ for all $t$, the contract reduces to a cost reimbursement contract. On the other hand, when $V'(t) = \alpha < 1$, we have the linear cost sharing contract. By allowing general cost sharing function $V'(t)$, our feasible contract options expand considerably.

To solve for the optimal choice of $V'(t)$, we note that WLOG we may assume $e(\pi) = 0$. Hence the optimal contract design problem reduces to

$$\min_{V'(t)} \left( \int_0^{\pi} V'(t)dt + \int_0^{\pi} \int_0^{n - e(n)} (1 - V'(t))f(n)dtdn \right)$$

$$= \min_{V'(t)} \left( \int_0^{\pi} V'(t)dt \right. $$

$$+ \int_0^{\pi} \left[ \int_{1+(\psi')^{-1}(V'(t))}^\pi f(n)dn \right] (1 - V'(t))dt$$

$$= \min_{V'(t)} \left( \int_0^{\pi} V'(t) dt \right. $$

$$\left. + P \left( n \geq t + (\psi')^{-1}(V'(t)) \right) (1 - V'(t)) dt \right)$$

For each $t$, the optimal choice of $V'(t)$ is

$$V'(t) := \arg\min_{x \in [0, 1]} \left( x + P(n \geq t + (\psi')^{-1}(x))(1 - x) \right).$$

Note that when $t = 0$, since $P(n \geq (\psi')^{-1}(x)) = 1$ for all $x$, the solution $V'(t) = 1$ is optimal. Similarly, when $t = \pi$, the solution $V'(t) = 0$ is optimal.

We use a few examples to demonstrate the viability of the proposed approach. Consider the case when $\psi'(t) = t$, and $n$ is uniformly distributed between 1 and 2. In a cost reimbursement contract, there is no incentive for the agent to reduce operational cost, and hence the expected cost of the operations to the principal is simply 1.5. On the other hand, for a combination of a fixed cost contract with $F = 1$ and a cost reimbursement contract, the agent will choose the fixed cost contract whenever

$$(n - 1) + 0.5 \leq 1, \ \text{i.e.,} \ n \leq 1.5,$$

and the cost reimbursement contract otherwise. The expected cost to the principal is then $(1 \times 1/2 + 1.75 \times 1/2) = 1.375$. We can do better using a nonlinear cost sharing function. Solving for the above optimal control problem, we have

$$V'(t) = 1 - \frac{t}{2}.$$

The corresponding lump sum payment is thus $K = 1$. It is easy to see that the expected payment for the principal in this case reduces to 1.333, which is the best possible solution in the class of incentive compatible, truth revealing and individual rational procurement mechanism. In fact, we can establish a more general result:

**Theorem 1.** When $\psi'(t)$ is linear, and $n$ is uniformly distributed, the optimal procurement contract under the class of incentive compatible, truth revealing and individual rational procurement mechanism is a nonlinear cost sharing contract.

**Proof.** See Appendix.
3 Logistics Outsourcing

We use the above to design the logistics outsourcing contract for the classical newsvendor problem. A firm produces a product at unit cost $c$ and sells at a retail price $p$ to a market with $n$ potential customers. The number of potential customers $n$ is a random variable decided by exogenous issues (e.g. season, recession) and uniformly distributed in $[n, \pi]$. Each customer has the same probability of making a purchase denoted by $\beta$.

Let $X_i$ denote the activity of the i-th potential customer. If he buys the product then $X_i = 1$ otherwise $X_i = 0$. Then we have $P(X_i = 1) = \beta$, and $P(X_i = 0) = 1 - \beta$ for every customer. Let $D(n)$ denote the market demand with $n$ potential customers, then $D(n) = \sum_{i=1}^{n} X_i$. We note that $D(n)$ is approximately normally distributed in $[0, n]$ if $n$ is large enough according to the law of large number.

$$D(n) \sim N(n\beta, \sqrt{n\beta(1-\beta)})$$

Given $n$, the firm decides how much to produce $q$ by solving the following newsvendor problem (Porteus 2002)

$$\max_{z} [(p-c)n\beta - \sqrt{n\beta(1-\beta)}\Gamma(z)]$$

where $z \equiv (q-n\beta)/\sqrt{n\beta(1-\beta)}$ and $\Gamma(z) \equiv p[\phi(z)-z\Phi]+ez$, and $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and cumulative distribution function respectively. The first term in the above objective function is the profit on the mean demand, and the second term is the expected cost of supply/demand mismatch (i.e. the newsvendor loss) with output quantity $q \equiv n\beta + z\sqrt{n\beta(1-\beta)}$

$$\sqrt{n\beta(1-\beta)}\Gamma(z) = (p-c)E_D[D-q]^+ + cE_D[q-D]^+$$

which is minimized at $z^* = \Phi^{-1}(\frac{E_C}{p})$, therefore the expected profit can be written as

$$\Pi = (p-c)n\beta - \sqrt{n\beta(1-\beta)}\Gamma(z^*)$$

The newsvendor problem is actually to minimize the newsvendor loss, which increase with the increase of uncertainty (i.e. the variance of $D(n)$). The firm outsources the demand forecasting function to the agent, a third-party analysis of market demand, in order to minimize his newsvendor loss with more accurate demand prediction. The first job of the agent is to find out the realized value of $n$. However, this information is not privy to the principal who knows that the $n$ is uniformly distributed in $[n, \pi]$. Besides finding out $n$, the agent is paid to provide additional effort $e$ to reduce the demand uncertainty. Given $n$ potential customers, the agent is going to interview $e$ of them before the selling season at a disutility of $\psi(e)$ to confirm whether these customers will buy the product or not. It is easy to know that the newsvendor loss reduces to $\sqrt{(n-e)\beta(1-\beta)}\Gamma(z^*)$. Note that when $e = 0$, the newsvendor cost is the same as the original newsvendor problem. On the other hand, when $e = n$, demand is confirmed without uncertainty and the profit on the mean demand realizes.

After interview, the agent will report to the principal the confirmed demand and suggested output quantity $q$. We assume the firm (the principal) can deduce the actual value of $n-e$ with suggested output $q$. But he cannot observe the effort $e$ and the original $n$. Now the principal is facing a problem how the outsourcing contract should like to induce! the optimal effort from the agent. Due to the special squareroot structure of the newsvendor loss, it is difficult to get either optimal contract or optimal linear cost sharing contract by the existing approaches (Laffont and Tirole 1986, Chu and Sappington 2007). However, only some small change on our approach allow us to achieve the optimal contract easily.

Consider a nonlinear cost sharing contract, which consists of a lump sum payment $F$ and share of the newsvendor loss from the principal $\sqrt{(n-e)\beta(1-\beta)}\Gamma(z^*) - V(n-e)$. We continue to use $V(t)$ and $V'(t) \in [0, 1]$ to denote the newsvendor loss borne by the agent and the marginal rate of cost sharing respectively, where $t = n-e$. However hereinafter $V'(t)$ is no longer the first order derivation of $V(t)$ but

$$V(t) = \int_0^t V'(s)ds(\sqrt{s\beta(1-\beta)}\Gamma(z^*))$$

$$= \beta(1-\beta)\Gamma(z^*)\int_0^t V'(s)\frac{s}{2\sqrt{s}}ds$$

$$= \delta \int_0^t V'(s)\frac{s}{2\sqrt{s}}ds$$

where $\delta \equiv \sqrt{\beta(1-\beta)}\Gamma(z^*)$

The optimal outsourcing contract is the incentive scheme which incurs the minimum expected payment from principal to agent while satisfies the constraints of participation, boundary condition and incentive compatibility as mentioned above.

The participation and incentive compatibility constraints ensure that for given $n$, the effort level that the agent would like to provide satisfies

$$e(n) := \arg \min_{e(n)} [\delta \int_0^{n-e(n)} V'(s)\frac{s}{2\sqrt{s}}ds + \int_{n-e(n)}^n \psi'(n-s)ds]$$

which means $\delta \frac{V'(n-e(n))}{2\sqrt{n-e(n)}} = \psi'(e(n))$, $\forall n \in [n, \pi]$
Furthermore, from boundary condition we have
\[ F = V(\pi - e(\pi)) + \psi(e(\pi)) \]
\[ = \delta \int_0^{\pi - e(\pi)} \frac{V'(s)}{2\sqrt{s}} ds + \int_{\pi - e(\pi)}^{\pi} \psi'(\pi - s) ds \]

The optimal contract design problem for the principal can be written as
\[ \min_{V'(t)} E_n[F + \int_0^{n-e(n)} (1 - V'(t)) d(\delta \sqrt{t})] \]
\[ = \min_{V'(t)} \left[ \int_0^{\pi} \frac{V'(t)}{2\sqrt{t}} dt + \int_{\pi - e(\pi)}^{\pi} \psi'(\pi - s) ds \right] + E_n(\delta \int_0^{n-e(n)} 1 - V'(t) (dt)), \]
where \( \frac{\delta V'(n-e(n))}{2\sqrt{n-e(n)}} = \psi'(e(n)), \forall n \)

WLOG we assume \( e(\pi) = 0 \). Hence the above optimal contract design problem reduce to
\[ \min_{V'(t)} \left[ \int_0^{\pi} \frac{V'(t)}{2\sqrt{t}} dt + \int_{\pi}^{\pi} \frac{n-e(n)}{2\sqrt{t}} (1 - V'(t)) d(\delta \sqrt{t}) dF(n) \right] \]
\[ = \min_{V'(t)} \left[ \int_0^{\pi} \frac{V'(t)}{2\sqrt{t}} dt + \int_0^{\pi} \left( \int_{t+(\psi^{-1}(\delta \frac{V'(t)}{2\sqrt{t}}))}^{\pi} dF(n) \right) \frac{1 - V'(t)}{2\sqrt{t}} dt \right] \]
\[ = \min_{V'(t)} \left[ \int_0^{\pi} \frac{V'(t)}{2\sqrt{t}} dt + P(n \geq t + (\psi^{-1}(\delta \frac{V'(t)}{2\sqrt{t}})) \frac{1 - V'(t)}{2\sqrt{t}) dt} \right] \]

For each \( t \), the optimal \( V'(t) \) is
\[ V'(t) = \arg \min_{x \in (0,1]} [x + P(n \geq t + (\psi^{-1}(\delta \frac{x}{2\sqrt{t}})) (1 - x)] \].

Note that when \( t = 0, V'(0) = 0 \cdot \psi(-1)(e(n)) = 0 \). And when \( t = \pi \), since \( P(n \geq \pi + (\psi^{-1}(\delta \frac{\pi}{2\sqrt{\pi}})) = 0, \)
the solution \( V'(\pi) = 0 \) is optimal. After getting the optimal \( V'(t) \) for each \( t \), \( e(n) \) can be determined by using
\[ \frac{\delta V'(n-e(n))}{2\sqrt{n-e(n)}} = \psi'(e(n)), \]
then \( F \) and \( V(t) \) can be derived from above equations. It is necessary to note that \( e(n) \) should be an integer.

We use a numerical example to give a more clear picture of the optimal contract achieving by this approach.

Consider the case when the probability of making a purchase of each customer \( \beta = 0.5 \), the unit cost \( c = 440 \), the selling price \( p = 1320 \), \( \psi'(x) = x \), and \( n \) is uniformly distributed in \([16, 36]\), then \( \delta = 240 \). Therefore by solving the above optimal control problem, we have
\[ V'(t) = \frac{1}{2} + \frac{1}{240} \sqrt{t}(16 - t) \]
The corresponding lump sum payment is thus \( F = 684 \).

The example also gave us some interesting intuition that the principal prefer to outsource the forecasting only if the unit cost and the retail price is high enough comparing with \( n \), which leads to a large possible newsvendor loss. Obviously, it is unnecessary for the principal to reduce the uncertainty if its impact on profit is trivial, either when the newsvendor loss is very small because of small unit cost and retail price, or when the newsvendor loss is hard to reduce due to a huge number of potential customers.

4 Extension and Advantage of the Approach

As illustrated in the above section, our approach can solve a problem that other approach may not able to solve. To give a further insight, it is necessary to discuss on some extensions of this approach.

The first extension is \( \psi(e) \) is concave. Although the concave disutility function has considerably wide applications in logistics industry, most of the preview literatures has a common assumption that the disutility function \( \psi(e) \) is convex and seldom discuss on concave disutility functions. This limitation may be derived from a reason: most of the existing contract design approach become invalid with a concave disutility function, for instance, Laffont’s approach fail to get an optimal contract with a linear disutility. Due to its general and simple framework, our approach is able to solve some of these problem. However, the incentive scheme is usually unnecessary if the disutility function is concave. Consider the system (consists of principal and agent) benefit \( e - \psi(e) \), which is a convex function when \( \psi(e) \) is concave. It is clear that the optimal effort level is either 0 or the \( \pi \), which means the effort level is predictable and observable without any incentive. This is another reason why literatures focus on convex disutility.

Another extension also shows that the optimal contracts can be more widely applied in practice with using our approach. When \( n \) follows a complex distribution instead of the uniform distribution, the optimal contract becomes difficult to achieve by following other approaches which have to deal with lots of differentiation and integration.
For example, the normal distribution widely used in practice is seldom considered in optimal contract theory. On the contrary, the theoretical idea that solve this kind of problem with our geometric approach is quite clear. No matter which distribution $n$ follows, we can ‘straighten’ it to a uniform distribution with the support $[n, \pi]$. The idea is to change the scale of $t$ axis for every unit length by transformation between the distribution functions, in order to ensure every unit length along the changed axis has the same density $f(n')$, where $n'$ denotes the new coordinate in the changed axis distinguished from the original coordinate $n$. We are going to cover this extension in our further research.

Comparing with the preview approach, our geometric approach has the following advantage. Truth revealing is a dominant strategy for the agent. Recall and rewrite the truth revealing condition as

$$n = \arg \max \frac{\partial F}{\partial \hat{n}} - V(\hat{n} - e(\hat{n})) - \psi(e(\hat{n}|n))$$

From the first order condition, we obtain

$$[V'(n - e(n)) - \psi'(e(n))](e'(n) - 1) = 0 \ \forall n$$

which coincides with the constraint derived from the individual rationality and incentive compatibility

$$V'(n - e(n)) = \psi'(e(n))$$

Therefore, it is no longer to require the agent to announce $n$, the payment can be easily computed by the observable variance.

The other distinguishing advantage of our approach is that it can achieve a great performance with a simple payment menus. It is obviously the performance of nonlinear contract is very close to optimal contract.

$$u(\hat{n}|n) = T(\hat{n} - e(\hat{n})) - (\hat{n} - e(\hat{n})) - \psi(e(\hat{n}|n))$$

Let $u(n|n) \equiv u(n|n)$, by following the first order condition, we have the incentive compatibility

$$u'(n) = -\psi'(e(n))$$

Therefore, the optimal scheme is the solution to

$$\min_e E_n[T(n - e(n))]$$

$$s.t. \quad u'(n) = -\psi'(e(n))$$

$$u(n) \geq 0, \forall n$$

By solving this optimal control problem, we have the optimal effort level

$$e(n) = \left\{ \begin{array}{ll} 2k - (n - \pi) & n \leq 2k + \pi \\ 0 & n \geq 2k + \pi \end{array} \right.$$ 

And the optimal payment from principal is

**Case 1.** $\Delta \leq 2k$, where $\Delta \equiv \pi - n$

$$T(n - e(n)) = n - e(n) + u(n) + \psi(e(n))$$

$$= \frac{(n - n)^2}{2k} - \frac{\Delta}{4k} + \pi - k$$

We rewrite the above equation by introducing $t \equiv n - e(n) = 2n - n - 2k$

$$T(t) = t - \left[ \frac{1}{2} + \frac{n}{4k} \right] t - \frac{t^2}{8k} + \left[ \frac{n^2}{8k} + \pi - \frac{n}{2} - \frac{\Delta^2}{4k} - \frac{k}{2} \right]$$

6 Conclusion

In this paper, we introduce a simple and refreshing geometric approach to study the optimal contract design and its application in logistics outsourcing issues, using a general cost sharing function. This approach simplifies the design of optimal contract and make some difficult problems solvable, which we show with a newsvendor problem. The nonlinear cost sharing contract yield by this approach consists of a fix lump sum payment $F$ and the sharing cost $(n - e - V(n - e))$, which is easy to implement in practice. Both theoretical proof and numerical example show that the nonlinear cost sharing contract can do better than the other simple schemes. In some case, it is the optimal contract. One aspect that we leave for future research is the case in which $n$ follows a non-uniform distribution. Although we proposed the idea how to solve this kind of problems in the extension section. However, detail theoretical proof and numerical analysis has to be completed in further research. With this extension, lots of other application in logistics and supply chain management can be developed. Another important question not completely answered in this paper is in which situation, the nonlinear contract is the optimal contract. Although we know nonlinear contract has great performance, we have to know the distance between optimal contract and nonlinear contract.
Let $F \equiv \frac{n^2}{8k} + \frac{n}{2} - \frac{\Delta^2}{8k} - \frac{k}{2}$ and $V(t) \equiv \left(\frac{1}{2} + \frac{n}{8k}\right)t - \frac{t^2}{8k}$, it is obvious that the payment scheme is a nonlinear cost sharing contract. Similarly, in case 2

**Case 2.** $\Delta \geq 2k$

$$T(n - e(n)) = \begin{cases} 
    n + \frac{(n-n)^2}{2k} & n \leq 2k + n \\
    0 & n \geq 2k + n 
\end{cases}$$

Rewrite it with $t$

$$T(t) = t - \left[\left(\frac{1}{2} + \frac{n}{4k}\right)t - \frac{t^2}{8k}\right] + \left[\frac{n^2}{8k} + \frac{n}{2} + \frac{k}{2}\right]$$

Let $F' \equiv \frac{n^2}{8k} + \frac{n}{2} + \frac{k}{2}$ and $V(t) \equiv \left(\frac{1}{2} + \frac{n}{8k}\right)t - \frac{t^2}{8k}$, it is a similar nonlinear cost sharing contract which only has a different lump sum payment with the one in case 1.

Q.E.D

**References**


